The Angular-Metric Traveling Salesman Problem

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Abstract

Motivated by applications in robotics, we formulate the problem of minimizing the total angle cost of a TSP tour for a set of points in Euclidean space, where the angle cost of a tour is the sum of the direction changes at the points. We establish the NP-hardness of both this problem and its relaxation to the cycle cover problem. We then consider the issue of designing approximation algorithms for these problems and show that both problems can be approximated to within a ratio of $O(\log n)$ in polynomial time. We also consider the problem of simultaneously approximating both the angle and the length measure for a TSP tour. In studying the resulting tradeoff, we choose to focus on the sum of the two performance ratios and provide tight bounds on the sum. Finally, we consider the extremal value of the angle measure and obtain essentially tight bounds for it. In this extended abstract we restrict our attention to the *planar* setting, but all our results are easily extended to higher dimensions.

1 Introduction

The traveling salesman problem (TSP) is one of the most widely studied combinatorial optimization problem, to the extent that there is an entire book [14] devoted to it. The importance of this problem is in large part due to the richness of its structure which has led to the development of a variety of general paradigms and techniques for dealing with optimization problems. We further explore this structure by formulating a natural version of the TSP problem that does not seem to have been studied earlier in the literature [14]. The new problem, called the Angle-TSP problem, seeks to minimize the total angle of a TSP tour for a set of points in Euclidean space, where the angle of a tour is the sum of the direction changes at the points. We establish the NP-hardness of this problem and the related cycle-cover problem, provide approximation algorithms for both problems, study their extremal properties, and give tight bounds on the trade-off between the angular and the length performance ratio achievable by a TSP tour.

Formally, we define the optimization problem **Angle-TSP** as follows: given a collection $P = \{p_1, \ldots, p_n\}$ of n points in Euclidean space, find a tour of the points in P with minimum total angle. For two edges (u, v) and (v, w) incident on a vertex v, we define the angle subtended by them as the absolute value of the change in the direction of motion when traveling from u to w via v (or, equivalently, when traveling from w to u via v) along these edges. For a path or a cycle on a subset of P, the total angle subtended is defined as the sum over all pairs of adjacent edges of the angle subtended by the pair. We also define an angular-metric version of the cycle cover problem (CCP) called **Angle-CCP:** given a collection $P = \{p_1, \ldots, p_n\}$ of n points in the Euclidean plane, find a cycle cover of the points in P subtending the minimum possible total angle. As will become clear shortly, unlike the length-metric cycle cover problem that is relatively easy, the Angle-CCP problem is as hard as the Angle-TSP problem itself.

Our main results are as follows. We show that the Angle-TSP and the Angle-CCP problem are both NP-hard. Given this, we turn our attention to designing approximation algorithms for these problems and show that both problems can be approximated to within a ratio of $O(\log n)$ in polynomial time. We also consider the problem of simultaneously approximating both the angle and the length measure for a TSP tour. In studying the tradeoff between the angle and the length metric, we choose to focus on the sum of the two performance ratios and provide tight bounds on the sum. Finally, we consider the extremal value of the angle measure and obtain essentially tight bounds for

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this too. In this extended abstract we restrict our attention to *planar* Angle-TSP but all our results are easily extended to higher dimensions.

Our original motivation was a problem posed by Guibas, namely, to find a "smooth" tour that visits a set of prescribed viewpoints of an object [10]. The smoothness criteria arises naturally in the context of nonholonomic robots [2, 3, 12]. Of late, there has been considerable interest in motion planning with "curvature constraints" (for example, see Agarwal, Raghavan, and Tamaki [1]). This is motivated by nonholonomic motion planning for steering-constrained robots that have stops on the steering mechanisms limiting the rate of change of direction [4, 5, 8, 11, 13]. Our problem involves a different notion of smoothness, but it could be used as a preprocessing step for planning curvature-constrained paths as suggested by Fraichard [8]. Our notion is of direct relevance in situations where a rotation by a robot is significantly more expensive than a translation. For instance, this would be the case for robots with high inertia, or where relatively large mechanical errors during rotation would necessitate error-detection and correction process via localization [12]. Also, there are applications in planning the trajectories of high-speed aircraft [7].

The Angle-TSP is a special case of TSP problems where the cost of an edge depends on the edges traversed earlier in the tour. Several applications can be modeled by such TSP problems. For example, the problem of scheduling orders in a production line can be modeled as a TSP problem where the vertices represent orders that have to be scheduled and edges represent set-up costs in moving from one order to another. In applications such as VLSI fabrication, steel mill scheduling, and chemical plants, the set-up cost of chemical cleansing and temperature control depend not only on the previous order, but on several previous orders. This translates to a TSP problem where the cost of an edge depends on previous edges. It should be noted that the Angle-TSP appears easier than this more general TSP problem, since an approximation algorithm for Angle-CCP would imply a similar approximation for Angle-TSP, but that is not the case in general.

Our work suggests many interesting and challenging directions for future work. Can our approximation ratio of $O(\log n)$ be improved to a constant, or be matched by a complementary hardness result? What can be said about other forms of trade-off such as the product of the angle and the length measures? It is also interesting to consider other natural variants of the basic Angle-TSP problem; for example, the variant where the maximum angle change at a vertex is bounded and the goal is to minimize the length measure.

The rest of this paper is organized as follows. We begin in Section 2 by showing that there are polynomial time $O(\log n)$-approximation algorithms for both Angle-TSP and Angle-CCP. In Section 3, we establish essentially tight bounds on the extremal cost of Angle-TSP. Building on the latter result, in Section 4, we establish tight bounds on the ability of a polynomial time algorithm to simultaneously approximate both the optimal Angle-TSP and the optimal Length-TSP. Finally, in Section 5, we sketch the proof of NP-hardness of Angle-TSP and Angle-CCP.

2 Approximation Algorithms for Angle-TSP and Angle-CCP

In this section, we describe an $O(\log n)$-approximation algorithm for the Angle-TSP problem on an instance P containing n points, with a running time polynomial in n. But first we need the following lemmas.

LEMMA 2.1. *Suppose there is a cycle C on a set of points P in the plane that subtends a total angle of α. Then, for any subset $P' \subset P$, there exists a cycle C' subtending a total angle at most α.*

The proof of Lemma 2.1 is straight-forward: using short-cuts to eliminate the points of $P \setminus P'$ from C, we obtain a cycle C' of no larger total angle by the triangle inequality for angles.

LEMMA 2.2. *There is a polynomial-time algorithm that finds a maximum cardinality path on the points in P that subtends a total angle at most π.*

Proof. We use dynamic programming. Fix some directed edge s in P and define the array A_s as follows: for each directed edge e in P and for $1 \leq k \leq n$, the array entry $A_s[e,k]$ contains the minimum angle subtended by a path of cardinality k starting with the edge s and ending at edge e; by convention, the entry is ∞ if there is no such path of cost at most π. We claim that the array entries for paths of cardinality k can be computed from the array entries for paths of cardinality $k-1$ in polynomial time, as follows: for any directed edge e and any edge f directed into the source point of e, let $\theta_{f,e}$ be the angle between f and e; then, $A_s[e,k] \leftarrow \min_f \{A_s[f,k-1] + \theta_{f,e}\}$, and $A_s[e,k]$ is set to ∞ if it exceeds π. To verify the correctness of this computation, observe that in a path with total angle at most π, neither vertices nor edges can repeat.

It is fairly easy to modify the preceding construction to store actual paths, rather than the angle subtended by them. The maximum cardinality path starting from the edge s with total subtended angle at most π can be determined by finding the largest k for which there exists an edge e such that $A_s[e,k]$ is finite. Repeating this for all choices of the starting edge s gives the desired result. It is easy to verify that this computation can be performed in polynomial time. □

We now establish the desired result.

THEOREM 2.1. *There is a polynomial-time algorithm which gives $O(\log n)$-approximation for Angle-TSP.*

Proof. Suppose that we are given an instance P consisting of n points $\{p_1, \ldots, p_n\}$ in the plane. Let τ^* be the minimum angle tour for P, where $\theta = \theta_{opt}(P)$ is the total angle change in the optimal cycle. Define $K = \lceil \theta/\pi \rceil$. It follows that the tour τ^* contains a subpath of cardinality at least n/K which subtends a net angle change at most π.

We now claim that for any instance P with n points with $\theta_{opt}(P) \leq K\pi$, it is possible to compute in polynomial time, a cycle cover $\mathcal{C} = \{C_1, \ldots, C_r\}$ for P with the following properties: $r = K \log n$; and, each cycle C_i subtends a total angle at most 3π. It can be verified that patching together any two cycles C and C' subtending total angles α and α' gives a single cycle C'' subtending an angle $\alpha'' \leq \alpha + \alpha' + 2\pi$. From this, it follows that repeatedly patching together the cycles in $\mathcal{C}$ gives a tour τ with $\theta(\tau, P) = O(\pi K \log n)$. Since $\theta_{opt}(P) = \Omega(\pi K)$, it follows that $R_\theta(\tau, P) = O(\log n)$, giving the desired result.

It remains to validate the polynomial-time construction of the claimed cycle cover. We do so by induction on n. The base case is trivial and is omitted. For the induction step, first observe that by Lemma 2.2, we can find a path of cardinality at least n/K in P with total angle at most π. Joining together the two endpoints of this path gives a cycle C_1 subtending a total angle at most 3π. Consider the instance P' with n' points obtained by deleting the points in C_1 from P. From Lemma 2.1, it follows that $\theta_{opt}(P') \leq \theta_{opt}(P) \leq K\pi$. Since $n' \leq n(1 - 1/K)$ and $K \log n' \leq K \log n - 1$, we obtain from the induction hypothesis that it is possible to recursively compute in polynomial time a suitable cycle cover for P' consisting at most $K \log n - 1$ cycles. Adding C_1 to this collection of cycles gives the desired cycle cover for P. □

We note that the same proof applies also to the Angle-CCP problem.

COROLLARY 2.1. *There is a polynomial-time algorithm which gives $O(\log n)$-approximation for* Angle-CCP.

3 The Extremal Cost of Angle-TSP

In this section, we address the question: What is the extremal cost of an optimal Angle-TSP as a function of n? While this question is of intrinsic interest in its own right, we will see in Section 4 that it also enables us to analyze the trade-off between the performance ratio for the angle and the length metrics.

Note that for any n points in the plane there is a solution for the Angle-TSP of cost $O(n)$. We improve this upper bound and show that for any n points in the plane there is a solution for the Angle-TSP of cost $O(n(\log\log n)^2/\log n)$. On the other hand, we prove that there exists an instance for which the optimal cost is $\Omega(n/\log n)$.

THEOREM 3.1. *For any n points in the plane, there exists a solution for the Angle-TSP problem of cost $O(n(\log\log n)^2/\log n)$.*

Proof. Our proof is constructive. For the proof we need the following definitions. We say that a rooted binary tree is *full* if each of its internal nodes has two children. Suppose that we are given a full rooted binary tree T with n' leaves and a 1–1 mapping from the leaves of T to a subset of the points (where each leaf is mapped to a point in the subset). We extend this mapping to the rest of the nodes of T in the obvious way: an internal node v is mapped to the set of points mapped to the leaves of the subtree rooted at v. (For consistency, assume that each leaf is mapped to a singleton set consisting of the point it is mapped to.) For a node v of T, let P_v denote the set of points to which it is mapped; also, let P_T be the set of points to which the root of T is mapped.

Given the mapping, we define a *center of gravity point* C_v for each node v as follows. For a leaf v the center is p, where $P_v = \{p\}$. Consider an internal node v of T, and let ℓ and r be the two children of v. Inductively, assume that C_ℓ and C_r are already defined. The center point C_v lies on the straight line connecting C_ℓ and C_r. Its distance from C_ℓ is $(|P_r|/|P_v|) \cdot dist(C_\ell, C_r)$, where $dist(C_\ell, C_r)$ is the distance between C_ℓ and C_r. Define the center of the tree T, denoted C_T, to be the center of its root.

We say that T is a *geometric tree with propagation* α of a point set P, if there is a 1–1 mapping from the leaves of T to a subset of P, such that for any internal node v of T of height at least two, $dist(C_\ell, C_r) \geq \alpha dist(C_{\ell\ell}, C_{\ell r})$, and $dist(C_\ell, C_r) \geq \alpha dist(C_{r\ell}, C_{rr})$, where $\ell\ell$ and ℓr are the children of node ℓ and $r\ell$ and rr are the children of node r, if such exist. (Since v is of height at least two, at least one child of v has two children.)

The proof is based on the following two lemmas.

LEMMA 3.1. *For any set of n points in the plane we can construct a geometric tree T with propagation* $\log n$ *and $|P_T| = n^{\Omega(1/\log\log n)}$.*

LEMMA 3.2. *If T is a geometric tree with propagation α with $|P_T| \geq m^d$, then we can construct a path containing at least m of the points in P_T such that the angular cost of each turn in the path is $O(1/d + 1/\alpha)$.*

The proofs of these lemmas are deferred to the full version of the paper. Here we concentrate on showing how they imply Theorem 3.1. Using Lemma 3.1 we construct a geometric tree T with propagation $\log n$ with

$$\begin{aligned} |P_T| &= n^{\Omega(1/\log\log n)} \\ &= \Omega\left[\left(\frac{\log n}{(\log\log n)^2}\right)^{\log n/(\log\log n)^2}\right]. \end{aligned}$$

Using Lemma 3.2 we find a path of length $\log n/(\log\log n)^2$ in P with angular cost $O(1)$, and remove the points of this path from P. We repeat this procedure until the number of points in P is reduced to $n/\log n$. The number of iterations is $O(n(\log\log n)^2/\log n)$. Finally, we connect all the remaining points in P using an arbitrary path, and stitch all the computed paths. The angular cost of the resulting tour is $O(n(\log\log n)^2/\log n + n/\log n) = O(n(\log\log n)^2/\log n)$. □

We now establish a lower bound on this extremal cost.

THEOREM 3.2. *There exists an instance P of Angle-TSP with n points in the plane such that the optimal cost for this instance is $\Omega(n/\log n)$.*

Proof. We describe a recursive construction that achieves this lower bound. Assume that $n = 2^k$, for some integer k. In the construction, we have $k = \log n$ levels of recursion. Define an instance of level i to be the instance constructed after i levels of recursion, and note that the final instance is the instance of level k. The instance of level 0 simply contains one point. The instance of level $i+1$ consists of two instances of level i at distance 4^i from each other such that the line connecting the centers of these two instances makes an angle of $i\pi/k$ with the X axis. We say that these two instances are *aligned* at angle $i\pi/k$. Note that the instance of level k has $2^k = n$ instances of level 0, and in general 2^{k-i} instances of level i.

Consider any solution to Angle-TSP on this instance. To prove the lower bound we show that the cost of traversing any three consecutive points in this solution is $\Omega(\pi/k)$. This readily implies the desired bound.

Fix any three consecutive points, and let i be the maximum level such that these three points do not belong to the same instance of level i. Clearly, such an i exists since the instance of level one consists of only two points. Note that these three points belong to the some single instance of level $i+1$. It is not difficult to verify that the cost of traversing points in the two instances is $\Omega(\pi/k)$, since: **(i)** the two instances of level i are aligned at angle $i\pi/k$; **(ii)** all the instances of lower levels are aligned at an angle that is at most $(i-1)\pi/k$; and, **(iii)** the distance between points in different instances of level is at least twice the distance between points in the same instance of level i. □

4 Simultaneously Approximating Length and Angle

In this section we discuss the possibility of *simultaneously* obtaining a good approximation to both the angle measure and the length measure for planar TSP. Fix an instance P and consider any tour τ for it. Denote by $\theta(\tau, P)$ the total angle subtended by τ, and by $\ell(\tau, P)$ the total length of τ. Further, let $\theta_{opt}(P)$ be the angle subtended by the optimal Angle-TSP solution for P, and $\ell_{opt}(P)$ be the total length of the optimal Length-TSP solution for P. Note that the optimal Angle-TSP and the optimal Length-TSP solutions for P need not be the same. We define the performance ratio of τ with respect to the two measures as:

$$R_\theta(\tau, P) = \frac{\theta(\tau, P)}{\theta_{opt}(P)} \qquad \text{and} \qquad R_\ell(\tau, P) = \frac{\ell(\tau, P)}{\ell_{opt}(P)}.$$

Our goal is to study the tradeoff between the two ratios $R_\theta(\tau, P)$ and $R_\ell(\tau, P)$ over all possible TSP solutions τ.

THEOREM 4.1. *There exists an instance P consisting of n points in the plane, such that for any tour τ on P,*

$$R_\theta(\tau, P) + R_\ell(\tau, P) = \Omega\left(\sqrt{n/\log n}\right).$$

Proof. Consider the following scenario in the plane. Fix $t = \sqrt{n \log n}$ and let $P_1, \ldots, P_t$ be a collection of identical instances described in the proof of Theorem 3.2 each with $\sqrt{n/\log n}$ points. Scale these instances so that the length of the path connecting the points in each instance is of unit length. We arrange these instances in a vertical stack so that they are perfectly aligned, and two successive instances in the vertical sequence are at a unit distance from each other. We refer to this entire set of points as the instance P.

We claim that: **(a)** $\ell_{opt}(P) = O(\sqrt{n \log n})$; and, **(b)** $\theta_{opt}(P) = O(\sqrt{n/\log n})$. To see (a), observe that by our scaling the optimal Length-TSP for any of the instances P_i has unit length; also, since the optimal Length-TSP tour can visit the instances in the vertical order, the total cost of traveling between the series of

$\sqrt{n \log n}$ instances is $O(\sqrt{n \log n})$. To see (b), observe that a set of $\sqrt{n/\log n}$ straight lines cover all points in P and these can be stitched together into a tour using $O(\sqrt{n/\log n})$ changes in the direction of travel.

Fix any tour τ for P. Consider any three consecutive points in τ. If these points are in the same instance P_i then, by the proof of Theorem 3.2, the *angular* cost of traversing them is $\Omega(1/\log n)$. If these points belong to at least two different instances then, because of the spacing between instances, the *length* cost of traversing them is $\Omega(1)$. Consider the $\lfloor n/2 \rfloor$ consecutive triples. We either incur the angular cost for at least half of them ($\geq \lfloor n/4 \rfloor$), or we incur the length cost for at least half of them. We conclude that $\theta(\tau, P) = \Omega(n/\log n)$ or $\ell(\tau, P) = \Omega(n)$. Combining this with the upper bounds on $\ell_{opt}(P)$ and $\theta_{opt}(P)$, we conclude that $R_\theta(\tau, P) + R_\ell(\tau, P) = \Omega(\sqrt{n/\log n})$. □

We can show that there is a polynomial-time algorithm that matches the lower bound to within a factor of $\log \log n$. (As a matter of fact for some values of $\theta_{opt}(P)$ the factor would be even smaller.) In this abstract, we prove the following marginally weaker upper bound. The tighter result is considerably more difficult to obtain and is stated without proof later in Theorem 4.3.

Theorem 4.2. *Given a set P of n points in the plane, there is a polynomial-time algorithm for constructing a tour τ on P such that: $R_\theta(\tau, P) + R_\ell(\tau, P) = O(\sqrt{n \log n})$.*

Proof. To find the desired tour we first compute an $O(\log n)$-approximation to the optimal Angle-TSP tour computed using the approximation algorithm described in Section 2. Denote the resulting tour by τ. We distinguish between two cases.

Case 1: $\theta(\tau, P) = \Omega(\sqrt{n \log n})$. In this case compute a constant approximation to the Length-TSP using, e.g., Christofides' approximation [6] for TSP. Denote the resulting tour by τ'. We claim that $R_\theta(\tau', P) + R_\ell(\tau', P) = O(\sqrt{n \log n})$. Clearly, $R_\ell(\tau', P) = O(1)$. Since $\theta(\tau, P) = \Omega(\sqrt{n \log n})$, we know that $\theta_{opt}(P) = \Omega(\sqrt{n/\log n})$. Also $\theta(\tau', P) = O(n)$. Hence $R_\theta(\tau', P) = O(\sqrt{n \log n})$.

Case 2: $\theta(\tau, P) = O(\sqrt{n \log n})$. In this case $R_\theta(\tau, P) = O(\log n)$. We claim that in $R_\ell(\tau, P) = O(\sqrt{n \log n})$. To see this we re-examine the approximation algorithm for the Angle-TSP problem. Recall that in this algorithm we find K' paths each of which subtends an angle of at most π that cover all the n points. Note that in our case $K' = O(\sqrt{n \log n})$. Below, we show that the length of each such path is a lower bound on $\ell_{opt}(P)$. Since there are $O(\sqrt{n \log n})$ such paths this implies that $R_\ell(\tau, P) = O(\sqrt{n \log n})$. (Note that the stitching of these paths costs $O(\sqrt{n \log n}\,\ell_{opt}(P))$.) Consider such a path Q with endpoints u and v. Clearly, the distance between u and v is a lower bound on $\ell_{opt}(P)$. We show that the length of Q is at most $2\sqrt{2}$ times this length. Partition Q into four segments, such that in each of the segments subtends an angle of at most $\pi/4$. We claim that the length of each such segment is $\sqrt{2}/2$ times the distance from u to v. Consider such a segment. The length of it is just the sum of the lengths of its edges. Consider the projections of all these edges on the straight line connecting u and v. The sum of the lengths of these projections is the distance between u and v. Since each edge has an angle of at most $\pi/4$ with this straight line, the projection of each edge is at least $\sqrt{2}/2$ of its length. □

The proof of the following tighter upper bound is much more involved and is deferred to the final version of the paper.

Theorem 4.3. *Given a set P of n points in the plane, there is a polynomial-time algorithm for constructing a tour τ on P for which $R_\theta(\tau, P) + R_\ell(\tau, P)$ is bounded as follows:*

(1) *if $\theta_{opt}(P) \leq \sqrt{\frac{n}{\log n}}$, then*

$$R_\theta(\tau, P) + R_\ell(\tau, P) = O\left(\theta_{opt}(P) \log\left(\frac{n}{\theta_{opt}(P)}\right)\right)$$

(2) *if $\sqrt{\frac{n}{\log n}} \leq \theta_{opt}(P) \leq \sqrt{\frac{n \log n}{\log \log n}}$, then*

$$R_\theta(\tau, P) + R_\ell(\tau, P) = O\left(\sqrt{\frac{n}{\log n}} \log \log n\right)$$

(3) *if $\theta_{opt}(P) \geq \sqrt{\frac{n \log n}{\log \log n}}$, then*

$$R_\theta(\tau, P) + R_\ell(\tau, P) = O\left(\frac{n}{\theta_{opt}(P)}\right).$$

5 The Intractability of Angle-TSP and Angle-CCP

In this section we prove that finding an optimal Angle-CCP is NP-hard. A similar reduction shows that the Angle-TSP problem is also NP-hard. We sketch the proof for the Angle-CCP problem, and then describe briefly how this proof can be modified to show the NP-hardness of the Angle-TSP problem.

Theorem 5.1. *The* Angle-CCP *and* Angle-TSP *problems are NP-hard.*

We provide a reduction to Angle-CCP from the One-in-Three 3SAT problem that is known to be NP-hard (Problem LO4 of Garey and Johnson [9]). The input to One-in-Three 3SAT is a set of m clauses $C_1, \ldots, C_m$, with each clause consisting of three literals from the set $\{u_1, \ldots, u_n, \overline{u}_1, \ldots, \overline{u}_n\}$, where $\{u_1, \ldots, u_n\}$ are boolean variables and $\{\overline{u}_1, \ldots, \overline{u}_n\}$ are their negations. The problem is to decide whether there exists a truth assignment to the variables such that each clause contains exactly one true literal. In the sequel we will use the notation N to denote $n + m$.

For an instance ϕ of the One-in-Three 3SAT problem, we construct a corresponding instance P of the Angle-CCP problem such that there exists a One-in-Three 3SAT solution to ϕ if and only if the corresponding instance P has a cycle cover of total angle at most T, for a value T to be specified later. We index the points in P by their coordinates along the X and Y axes.

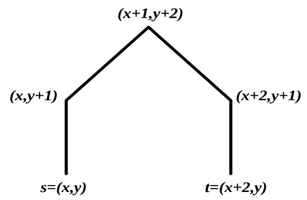

Figure 1: A U-turn Gadget in Direction $+Y$

For the reduction we need a gadget that we call a "U-turn gadget," although "V-turn gadget" would perhaps be a more appropriate name for it. This consists of points on two straight lines and on two diagonals connecting those lines as shown in Figure 1. Each such gadget is characterized by its *endpoints* and its *direction*. Intuitively, a gadget pointing in the direction $+Y$ with endpoints $s = (x, y)$ and $t = (x+2, y)$ consists of a set of points in the plane such that in order to cover them efficiently we have to make a U-turn after entering at s (t) in the direction $+Y$ and exiting at t (respectively, s) in the direction $-Y$. To achieve this goal, the gadget contains $12N^5$ points. Of these, $10N^5$ points are spread uniformly on the two unit-length intervals from (x, y) to $(x, y+1)$, and from $(x+2, y)$ to $(x+2, y+1)$. The remaining $2N^5$ points are spread uniformly along the two diagonals from $(x, y+1)$ to $(x+1, y+2)$ and from $(x+2, y+1)$ to $(x+1, y+2)$.

We are now ready to define the instance P of Angle-CCP. In P, we have nine points corresponding to each clause C_i. Suppose that clause C_i consists of the literals u_a,$\overline{u}_b$, and u_c. The points corresponding to this clause are: (N^2i, N^2a), (N^2i, N^2b), (N^2i, N^2c+1), (N^2i+1, N^2a+1), (N^2i+1, N^2b+1), (N^2i+1, N^2c+1), (N^2i+2, N^2a+1), (N^2i+2, N^2b), and (N^2i+2, N^2c). (See Figure 2.)

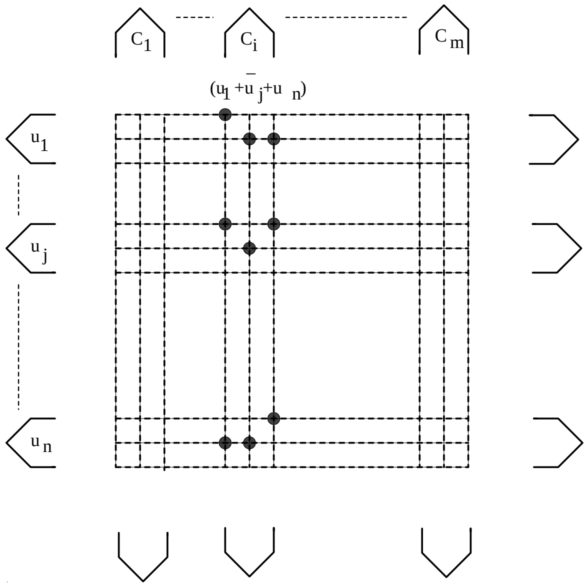

Figure 2: Illustrating the Reduction

Let us explain the reasoning behind this assignment of points. We allocate three "columns" to each clause, where the clause C_i is associated with the columns with X-coordinates N^2i, N^2i+1, and N^2i+2. Similarly, we allocate three "rows" to each variable, where the variable u_j is associated with the rows with Y-coordinates N^2j, N^2j+1, and N^2j+2. In addition, we allocate the row N^2j to the literal u_j, and the row N^2j+1 to the literal $\overline{u}_j$. (We assume that $N = (n + m)$ is large enough.)

Let us fix our attention on a clause C_i and the three columns associated with it. For $j \in \{1, 2, 3\}$, in the jth column associated with this clause, we have the points in its intersection with the rows corresponding to the jth literal in the clause and the complements of the other two literals. In addition, we have the points (N^2i+3, N^2j+2), for $j = 1, \ldots, n$, and $i = 1, \ldots, m$. We call all the points defined so far *grid points*.

Finally, we add $2N$ U-turn gadgets. For each variable u_j, $1 \le j \le n$, we have two U-turn gadgets – one with endpoints $(-N^5, N^2j)$ and $(-N^5, N^2j+2)$ pointing in the direction $-X$ (the "left" gadget), and another with endpoints $(N^5 + N^2m, N^2j)$ and $(N^5 + N^2m, N^2j+2)$ pointing in the direction $+X$ (the "right" gadget). Similarly, for each clause C_i, $1 \le i \le m$, we have two U-turn gadgets – one with endpoints $(N^2i, -N^5)$ and $(N^2i+2, -N^5)$ pointing in the direction $-Y$ (the "bottom" gadget), and another with endpoints $(N^2i, N^5 + N^2n)$ and $(N^2i+2, N^5 + N^2n)$ pointing in the direction $+Y$ (the "top" gadget).

Our goal now is to show that the instance of the One-in-Three 3SAT has a satisfying assignment if and only if the corresponding instance of Angle-CCP has a cycle cover of total angle at most $T = 2\pi N + 4N^{-4}$.

LEMMA 5.1. *If there exists a One-in-Three truth assignment satisfying ϕ, then there is a cycle cover for P of total angle at most T.*

Proof. We will construct a cycle cover that has N cycles, one per clause and one per variable. Consider a clause C_i and suppose that the literal that satisfies C_i is the first literal; the other cases are similar. The cycle corresponding to C_i covers the points on the two U-turn gadgets of C_i and also the points in two of the columns assigned to C_i. These are the columns that do not contain the point intersecting the row assigned to the literal satisfying C_i, i.e., the cycle covers the columns with X-coordinates N^2i+1 and N^2i+2.

We bound the angle of the cycle starting from point $(N^2i, -N^5)$, going counter-clockwise. The angle cost of covering the points of the bottom gadget exiting at $(N^2i+2, -N^5)$ is π. From here, we can cover all the points in column N^2i+2 without incurring any angle cost. Then, the angle cost of covering the top U-turn gadget entering at (N^2i+2, N^5+N^2n) and exiting at (N^2i, N^5+N^2n) is again π. From this point we traverse to point (N^2i+1, N^2n) at an angle cost of (approximately) $2N^{-5}$ and then, without incurring any further angle cost, cover all the points in column $N^2i{+}1$. Finally, we traverse from $(N^2i+1, 0)$ to $(N^2i, -N^5)$ at an angle cost of (approximately) $2N^{-5}$. Overall, the angle of the cycle is $2\pi + 4N^{-5}$.

The remaining points are covered by the cycles corresponding to the variables. Consider a variable u_j, and suppose that the assignment of u_j is TRUE; the other case is similar. The cycle corresponding to u_j covers the points on the two U-turn gadgets of u_j. It also covers the points in two of the rows assigned to u_j, viz., the row N^2j corresponding to the literal u_j and row N^2j+2.

We now bound the angle of the cycle starting from point $(-N^5, N^2j+2)$, going counter-clockwise. The angle cost of covering the points of the left gadget exiting at $(-N^5, N^2j)$ is π. From here, we cover all the points in row N^2j at no additional angle cost. Then, the angle cost of covering the right U-turn gadget entering at $(N^5{+}N^2m, N^2j)$ and exiting at $(N^5{+}N^2m, N^2j{+}2)$ is again π. From this point, we cover all the points in row N^2j+2 at no additional angle cost. Overall, the angle cost is 2π. Note that in case u_j is assigned FALSE, the total angle is $2\pi + 4N^{-5}$.

Summing over all the $N = (n{+}m)$ cycles, we obtain that total angle is bounded by $T = 2\pi N + 4N^{-4}$. □

The following lemmas help us establish the reverse direction.

LEMMA 5.2. *The angle cost of covering only the U-turn gadgets in the P is $2\pi N$. The cover that achieves this has N cycles, each of angle 2π and each of which covers a pair of gadgets corresponding to either a clause or a variable. Moreover, any other cover that traverses the points in a different order has an angle cost at least $2\pi N + \Omega(N^{-3})$.*

Proof. It can be verified that the cover with N cycles in which each pair of gadgets corresponding to either a clause or a variable is covered by one cycle indeed achieves an angle cost of $2\pi N$. We now prove that this is the only cover of angle cost $2\pi N + o(N^{-3})$. Clearly, any cycle cover of this angle cost has at most N cycles. Consider such a cover $\mathcal{C}$.

Recall that each gadget contains two straight line intervals. We call these lines the *spokes* of the gadget. We have horizontal and vertical spokes. Each spoke has two endpoints: *internal* and *external*. The internal endpoint is the one closer to the grid points. For simplicity, consider first only cycle covers that cover all the points in each such spoke consecutively; i.e., when we follow a cycle in $\mathcal{C}$, the first point of a spoke that we encounter is an endpoint, and starting from it we cover all the spoke points ending at the other endpoint.

We now prove that any cycle cover $\mathcal{C}$ with the above property that covers only the spoke points must have an angle cost of at least $2\pi N$. Clearly, the cover described in the lemma achieves this angle cost. To show this we prove that the "average" angle cost of covering each spoke is at least $\pi/2$, since there are $4N$ spokes this implies the bound. Consider a cycle $C \in \mathcal{C}$. Note that by our assumption, whenever C covers a horizontal (vertical) spoke it subtends an angle 0 or π (respectively, $\pi/2$ or $3\pi/2$) with the $+X$ direction. There are two cases: either C covers only spokes of the same orientation (horizontal or vertical), or C covers both horizontal and vertical spokes. In the first case it is easy to see that the angle cost of covering k spokes of the same orientation is at least $\lceil k/2 \rceil \pi$. Thus the average angle cost is at least $\pi/2$, assuming $k > 1$.

Consider a cycle C that covers alternate runs of horizontal and vertical spokes. For each such run we distinguish whether if starts (ends) in an internal or external endpoint. The following properties are easy to verify.

1. The angular cost subtended when moving from an external endpoint of a vertical (horizontal) spoke to an external endpoint of a horizontal (respectively, vertical) spoke is at least $3\pi/2 - o(N^{-2})$.

2. The angular cost subtended when moving from an internal endpoint of a vertical (horizontal) spoke to an internal endpoint of a horizontal (respectively, vertical) spoke is at least $\pi/2 - o(N^{-2})$.

3. The angular cost subtended when moving from an internal endpoint of a vertical (horizontal) spoke to an external endpoint of a horizontal (respectively, vertical) spoke, or vice versa, is at least $\pi - o(N^{-2})$.

4. The angular cost subtended while covering k spokes of the same orientation by a *path* that starts and ends in external endpoints is at least $\lceil (k-2)/2 \rceil \pi$. (Note that this is only the angle cost of the path without including the angle cost of reaching the first endpoint and leaving the last.)

5. The angular cost subtended while covering k spokes of the same orientation by a *path* that starts and ends in internal endpoints is at least $\lceil k/2 \rceil \pi$.

6. The angular cost subtended while covering k spokes of the same orientation by a *path* that starts in an internal endpoint, and ends in an external endpoint, or vice versa, is at least $\lceil (k-1)/2 \rceil \pi$.

These observations imply that the average angular cost of a spoke in such a cover is more than $\pi/2$. We conclude that to achieve a cover $\mathcal{C}$ of angle cost at most $2\pi N$ all the cycles must cover only spokes of the same orientation and in this case the cost of the cover is at least $2\pi N$.

We now show that the same holds even if we remove the assumption that spokes are traversed consecutively. For each spoke ℓ consider the interval of length N^{-3} starting at its external endpoint. This interval consists of $5N^2$ points. Hence, there is a cycle $C \in \mathcal{C}$ that covers at least $5N$ of these points; we say that this C is the *main cycle* for the spoke ℓ. We claim that C must traverse at least two points of the interval one after the other. This is because each time a point not in the interval is traversed between two points in the interval, the cycle C subtends an angle cost of $\pi - o(N^{-2})$. Note that in case ℓ is horizontal (vertical), C subtends an angle 0 or π (respectively, $\pi/2$ or $3\pi/2$) with the $+X$ direction. Among the points traversed consecutively, let the external point be the one closest to the external endpoint of ℓ, and the internal point be the one closest to the internal endpoint. It is not difficult to verify that the same arguments we used above would hold if we let the main cycle of ℓ play the role of the cycle that traverses ℓ (consecutively), and the internal and external points of the interval play the role of the internal and external endpoints.

Recall that each gadget contains also two diagonal lines. To achieve the desired angle cost, it must be the case that all points on a diagonal are covered by the main cycle of the spoke touching it, and that these points are traversed either just before or just after the points of the touching spoke are traversed. To see this, recall that every cycle covers at least two spokes and that all its covered spokes are of the same orientation. Consider a such a cycle, and assume that it covers some points on a diagonal line, between the traversal of two spokes not touching it. It is easy to see that in this case the additional angular cost subtended is at least $\pi/2$. This would make the average angular cost of a spoke $\pi/2 + O(N^{-1})$; a contradiction. Now, consider the leftmost gadget on the top. The cycle that covers the points on the right diagonal of this gadget must be the main cycle of its left spoke, otherwise its angle cost would be too large, following from arguments similar to the ones given above. Hence, we have that both spokes of this gadget have the same main cycle, which covers also the points on its two diagonals. Similarly, both spokes of the bottom leftmost gadget have the same main cycle, which covers also the points on its two diagonals. Moreover, it must be the case that the same cycle covers both gadgets. Otherwise, the cost of the cycle would be $k\pi + \Omega(N^{-3})$.

In the same way we can show that both gadgets corresponding to the second clause are covered by the same cycle, and so on. It follows that in order to achieve the desired angle cost, no cycle covers gadgets corresponding to more than one clause. In a similar fashion, it can be shown that the gadgets corresponding to a variable must be covered by the same cycle, and that no cycle covers the gadgets corresponding to more than one variable. We conclude that the only cycle cover that achieves the desired cost is the one with N cycles, each of which covers a pair of gadgets corresponding to either a clause or a variable. □

LEMMA 5.3. *If there exists a cycle cover P of total angle at most T, then there is a One-in-Three truth assignment satisfying ϕ.*

Proof. Suppose that there exists a cover of the Angle-CCP instance of angle cost at most $T = 2\pi N + 4N^{-4}$. Since it covers all the U-turn gadgets and its cost is bounded by $2\pi N + o(N^{-3})$, it must be that this cover consists of N cycles each of which covers the two gadgets corresponding to either a clause or a variable. Consider the cycle that covers the U-turn gadgets of a clause C_i. In addition to the points of the two gadgets, this cycle can cover at most two of the columns corresponding to C_i. Otherwise, the cost of the cycle would be $2\pi + \Omega(N^{-3})$. (Note that covering more than one column in a path connecting the two gadgets would incur a cost of $\Omega(N^{-3})$.) Similarly, the cycle that covers the U-turn gadgets of a variable u_j can additionally cover at most two of the rows corresponding to u_j. One of these rows must be row N^2j+2, since none of the other cycles can cover the points on this row within the angle bounds.

It follows that for each clause C_i, all the grid points in one of its columns are covered by cycles corresponding to variables. Recall that such a column consists of points in the rows corresponding to one literal $l_j \in \{u_j, \overline{u}_j\}$ in the clause, and to the complements of the other two. Assign a TRUE value to l_j, and assign a FALSE value to the other two literals in C_i. Repeating this for all clauses implies that in the resulting truth assignment, for each clause we have exactly one literal that is TRUE. We claim that this assignment is consistent, i.e., no variable is assigned both TRUE and FALSE. This is because a literal is assigned the value TRUE only if its corresponding row is covered by the cycle corresponding to its originating variable. However, a cycle corresponding to a variable u_j can cover either the row corresponding to the literal u_j or the row corresponding to the literal $\overline{u}_j$, but not both. This implies that ϕ has a One-in-Three truth assignment. $\square$

We briefly sketch the modifications required to prove the NP-hardness of the Angle-TSP problem. Again, the reduction is quite similar and is from the One-in-Three 3SAT problem. Given an instance ϕ of the One-in-Three 3SAT problem, the corresponding instance of the Angle-TSP problem would include the same grid points and the left/top U-turn gadgets as before. In addition, in the "right" and the "bottom" we add U-turn gadgets of a different size that are used to "stitch" the rows (columns) corresponding to adjacent variables (respectively, clauses). Finally, we add two other gadgets: one to connect the bottom end of the rightmost column with the right end of the bottom row, and another to connect the bottom end of the leftmost column with the right end of the top row. It can be shown that there exists a tour of total angle cost at most T if and only if there is a One-in-Three truth assignment satisfying ϕ.

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