On Wireless Spectrum Estimation and Generalized Graph Coloring

Sanjeev Khanna Krishnan Kumaran
Bell Laboratories, Lucent Technologies
Murray Hill, NJ 07974
{sanjeev,kumaran}@lucent.com

Abstract— **We address the problem of estimating the spectrum required in a wireless network for a given demand and interference pattern. This problem can be abstracted as a generalization of the graph coloring problem, which typically presents additional degree of hardness compared to the standard coloring problem. It is worthwhile to note that the question of estimating the spectrum requirement differs markedly from that of allocating channels. The main focus of this work is to obtain strong upper and lower bounds on the spectrum requirement, as opposed to the study of spectrum allocation/management.**

While the relation to graph coloring establishes the intractability of the spectrum estimation problem for arbitrary network topologies, useful bounds and algorithms are obtainable for specific topologies. In the first part of this work, we establish some new results regarding generalized coloring, which we use to derive tight bounds for specific families of graphs. The latter part is devoted to the hexagonal grid topology, a commonly used topology for wireless networks. We design efficient algorithms that exploit the geometric structure of the hexagonal grid topology to determine upper bounds on the spectrum requirement for arbitrary demand patterns. The slack in our upper bounds is estimated by analyzing subgraphs with specific properties. While we consider the worst-case demand patterns to evaluate the performance of our algorithms, we expect them to perform much better in practice.

Keywords— **Wireless Spectrum Estimation, Channel Assignment, Graph Coloring.**

I. Introduction

The subject of channel assignment in wireless systems is a widely studied one (for instance, see [KN96] for a comprehensive survey). Most of these studies address the issue of how a given block of spectrum can be managed efficiently in a dynamic network. Various fixed channel assignment (FCA) and Dynamic channel assignment (DCA) schemes have been proposed and analyzed in this context. The question of bounding the capacity/performance measures of the system for specific proposed channel assignment algorithms has received considerable attention, and capacity improvements using DCA are becoming increasingly common.

Our focus in this paper is on a rather different issue, albeit one of considerable importance to the design of wireless networks. We address the issue of bounding spectrum requirement given the projected demands at each cell and the structure of the co-channel and adjacent-channel interference in the environment. Our objective is not to determine a fixed assignment, but to use a given projected demand to estimate the spectrum requirement. With the increasing success of DCA algorithms, the spectrum may then be managed efficiently to provide high capacity and adaptivity. In comparison to the extensive work that has been done on channel assignment schemes, relatively little work exists on the issue of bounding the spectrum requirements. Much of the previous work, in our opinion, has approached this problem either with too much generality to obtain strong bounds [Hal80], [Gam86], or has restricted itself to simple topologies [GGS93](such as a row of base-stations) and/or simple demand patterns such as uniform demands [Gam82], [ZY89]. In contrast, this work makes a first attempt (to the best of our knowledge) to give strong upper and lower bounds on the spectrum requirement on moderately complex topologies. In particular, we derive non-trivial bounds for the widely used regular hexagonal grid topology with nearest and next-to-nearest neighbor interference.

Perhaps the main reason for the relatively low attention given to this problem is its relation to the notoriously difficult problem of graph coloring. Frequency assignment is a generalized version of the conventional coloring problem, as explained later, and carries an additional dimension of complexity as compared to the standard graph coloring. Even in simple cases such as cycles, where the coloring problem is trivially solved, non-unit demands generally require examination of many more independent sets.

However, as we will see later, having non-unit demands generally permits increased re-use that makes the generalized chromatic number behave very differently from the standard chromatic number even on simple topologies. In particular, when the demand is uniformly scaled up, the generalized chromatic number turns out to be closely related to another intractable parameter, namely, the *fractional chromatic number* of the graph, which is generally smaller than the chromatic number.

Broadly speaking, this paper is organized into four parts as described below:

Part I: It consists of Sections 2, 3 and 4, where we define the basic terminology, formalize our problem and establish connection to graph-theoretic parameters that relate to our problem. We explore an LP formulation of the problem, which is useful in exposing the relationship between generalized coloring and fractional coloring.

Part II: This part, confined to Section 5, focuses on graph topologies for which the spectrum estimation problem can be exactly solved in polynomial time. We start with simple examples such as bipartite graphs and move on to analyze some non-trivial topologies for generalized coloring, such as the odd cycles and their complements. Odd cycles play a very fundamental role in the later part of our work where they help determine the extent to which the spectrum requirements deviate from a well-known lower bound.

Part III: It comprises of Section 6 and focuses on determining a new lower bound on the spectrum requirement. Interestingly, we show that our new bound can be derived by two strikingly different approaches: one based on a graph-theoretic argument and the other using a linear programming approach. We demonstrate the strength of this bound by applying it to a simple graph that can be embedded in the hexagonal grid topology.

Part IV: This part, contained in Sections 7 and 8, is the primary objective of this work. Our focus in this part is to investigate the widely used hexagonal grid topology under nearest and next-to-nearest interference patterns. The results of previous sections, although used here, are in themselves inadequate to provide reasonable bounds for this problem, as they are oblivious to the rich geometric structure of this graph. Using the inherent geometry of this topology, we are able to establish new non-trivial upper bounds on the spectrum requirements in this case. Our algorithms are simple and can be implemented efficiently.

We conclude with a brief discussion in Section 9.

II. Problem Formulation and Statement

Consider a wireless network $\mathcal{N}$ consisting of a collection of n base-stations in a planar region; associated with each base-station is a *cell* which represents its area of service, say a hexagon in the plane. We say that two base-stations are *interfering* if their associated cells are close enough to interfere while using the same frequency. Hence any allocation of frequencies in the network must not assign the same frequencies to a pair of interfering base-stations. Based on the traffic demand within a given cell, one may estimate the number of frequencies needed to satisfy the total demand associated with the cell. Then, informally speaking, the *wireless spectrum estimation* problem is to determine the minimum number of frequencies needed such that each base-station can be assigned the needed number of frequencies and no pair of interfering base-stations share a frequency in common. To define our problem more formally, we introduce the notion of *interference graph* of a wireless network $\mathcal{N}$ as below:

Definition 1 (Interference Graph) Given a wireless network $\mathcal{N}$, its interference graph $G = (V, E)$ is defined as follows:

- there is a one-to-one correspondence between the nodes in V and the cells of the network $\mathcal{N}$, and
- there is an edge $(i, j) \in E$ if and only if the cells corresponding to nodes i and j interfere with each other.

The Wireless Spectrum Estimation Problem: We may now formalize the problem of our interest as follows:

Input: The interference graph G of a wireless network $\mathcal{N}$, and a demand vector $\mathbf{d} \in \mathcal{Z}^n$ indicating the demand at each base-station of $\mathcal{N}$ (alternatively, each vertex of G).

Goal: Estimate the minimum number of frequencies required to meet all demands such that no two interfering base-stations are assigned the same frequency.

III. Preliminaries

Prior to formalizing our problem as a generalized coloring problem, we review some notation and basic graph-theoretic definitions that are relevant to our work. For a more comprehensive treatment of the concepts mentioned here, the reader should refer to a standard graph theory or graph coloring text(for instance, see [Wes96] for general graph theory and [JT95] for graph coloring).

Given a graph $G = (V, E)$, we define its *chromatic number* to be the smallest number of colors needed such that every vertex is assigned a color and no two vertices connected by an edge receive the same color. The *clique number* of G is defined as the largest subset of vertices such that any two of them are connected by an edge in G. Similarly, the *independence number* is of G is defined to be the largest subset of vertices such that no two of them are connected by an edge in G. We use $n(G)$, $\chi(G), \omega(G)$, and $\alpha(G)$ to denote the number of vertices, *chromatic number*, *clique number*, and the *independence number* of G, respectively. When the underlying graph G is clear from the context, we would abbreviate our notation to n, χ, ω and α, respectively.

A class of graphs of particular interest to us is the class of *perfect graphs.*

Definition 2 (Perfect Graph) A graph G is perfect if and only if for every induced subgraph $G' \subseteq G$, $\chi(G') = \omega(G')$.

In other words, it is possible to color a perfect graph with the minimum possible number of colors. Moreover, such a coloring can be found in polynomial time [Lov72].

The above definitions pertain to unit demands, i.e., each vertex is assigned exactly one color. The problem of frequency assignment requires that a specified number of colors d_i, which we call the *demand*, be assigned to vertex i, so that interfering vertices (base-stations) are not assigned the same frequency[1]. A natural generalization of the parameters chromatic number, clique number and independence number, can then be made as follows. Suppose we are given a demand vector $\mathbf{d} \in \mathcal{Z}_+^n$ on the graph G, with the i_{th} component of $\mathbf{d}$ (d_i), indicating the demand of node i. We define:

Definition 3 (Generalized Chromatic Number) The generalized chromatic number of a graph G with an associated demand vector $\mathbf{d}$, denoted by $\chi^d(G)$, is defined to be the minimum number of colors needed to assign d_i colors to node i where $i \in [1..n]$ such that no adjacent pair of nodes shares a color.

Definition 4 (Generalized Clique (Independence) Number) The generalized clique number (independence number) of a graph G with an associated demand vector $\mathbf{d}$, denoted by $\omega^d(G)$ $(\alpha^d(G))$, is defined as $\max_{S\in\mathcal{F}}\{\sum_{i\in S} d_i\}$ where $\mathcal{F}$ denotes the set of cliques (independent sets) in the graph G.

[1] In this paper, we use the words *color* and *frequency* synonymously, with a historically motivated bias towards the former.

The following proposition is immediate from the above definitions.

Proposition 1: For any graph G with an associated demand vector $\mathbf{d}$,

$$\max\{\omega^d(G), \frac{\sum_i d_i}{\alpha(G)}\} \leq \chi^d(G) \leq (\max_i d_i)\chi(G).$$

Vertex-Replication Construction: We observe that the following construction reduces the generalized coloring problem to the conventional coloring problem with unit demands:

Proposition 2: Given a graph G with an associated demand vector $\mathbf{d}$, consider the graph G' obtained by replacing each vertex v_i of the graph $G(V, E)$ with demand d_i by d_i independent cliques of size d_i each, with each vertex in this set having edges to the same vertices as v_i in G. Then any valid color assignment in G induces a valid coloring on the graph G': assign the same color to corresponding vertices in the d_i cliques that replace v_i.

The above simple construction has several interesting implications that we discuss in later sections. In particular, it helps in deriving a new lower bound and also helps generalize Lovasz's results on perfect graphs to generalized coloring.

IV. LP Formulation and the Fractional Chromatic Number

On any general graph $G(V, E)$, one may in principle find the set $\mathcal{F}$ of all independent sets in the graph. One may then formulate the problem of finding an allocation that uses a minimum number of colors, as an Integer Linear Program(ILP). We assign an integer number of colors n_S to the independent set S, so as to

Minimize $N = \sum_{S \in \mathcal{F}} n_S$

Subject to:

$$\sum_{S \in \mathcal{F} \wedge v \in S} n_s \geq d_v \quad \forall \quad v \in V; \tag{1}$$

where $\mathcal{F}$ is the set of all independent sets in G.

Observe that if $d_v = 1$ for all v and n_S is restricted to be $\{0, 1\}$, the solution to the ILP is precisely the chromatic number. Since the set of all independent sets may be exponentially large (it may be as large as the power set of V), the above formulation is exponential in the input size. Even otherwise, the problem of solving integer linear programs is intractable in general. For special kinds of graphs and demand patterns, one may attempt to heuristically restrict the number of independent sets used in the optimization and the ILP approach may be rendered tractable. However, our interest in invoking this formulation here stems from the fact that it allows us to introduce an important parameter, namely the *fractional chromatic number*, obtained by relaxing the integrality constraints.

Definition 5 (Fractional Chromatic Number) The fractional chromatic number of a graph $G = (V, E)$, denoted $\chi^*(G)$, is defined to be the solution to the following linear program:

Minimize $\sum_{S \in \mathcal{F}} n_S$

Subject to:

$$\sum_{S \in \mathcal{F} \wedge v \in S} n_S \geq 1 \quad \forall \quad v \in V; \tag{2}$$

$$0 \leq n_S \leq 1$$

where $\mathcal{F}$ is the set of all independent sets in G.

Thus $\chi^*(G) \leq \chi(G)$. In the remainder of this section, we point out a curious relationship between the fractional and the generalized chromatic number. Let 1 denote the n-dimensional vector $[1, 1, ..., 1]$. Then the following theorem states the relationship between these two parameters:

Theorem 1: For any finite graph G,

$$\lim_{t \to \infty} \frac{\chi^{t\mathbf{1}}(G)}{t} = \chi^*(G) \tag{3}$$

Proof: Given an n-vertex graph G, first observe that for any integer $t \geq 1$, $\chi^*(G) \leq \chi^{t\mathbf{1}}(G)/t$ since by assigning a weight of $1/t$ to each independent set used in obtaining $\chi^{t\mathbf{1}}(G)$, we can obtain a fractional coloring of cost no more than $\chi^{t\mathbf{1}}(G)/t$. On the other hand, consider an optimal fractional coloring of G. It can be represented by a 2^n-dimensional vector $[n_1^*, n_2^*, ..., n_{2^n}^*]$ since a n-vertex graph can have at most 2^n distinct independent sets. Consider the following natural way to convert this fractional coloring to a solution for our generalized chromatic number problem: multiply each component by t and round upwards. The total number of colors used is bounded as below:

$$\sum_{i=1}^{2^n} \lceil t n_i^* \rceil \leq t\chi^* + 2^n.$$

Thus $\chi^{t\mathbf{1}} \leq t\chi^* + 2^n$. Therefore,

$$\lim_{t \to \infty} \frac{\chi^{t\mathbf{1}}(G)}{t} \leq \lim_{t \to \infty} \frac{t\chi^* + 2^n}{t} = \chi^*.$$

■

From the above proof, a careful reader may note that not only the quantity $\chi^{t\mathbf{1}}(G)/t$ converges to χ^*, each component of $\chi^{t\mathbf{1}}(G)$ converges to the correponding component of $[tn_1^*, tn_2^*, ..., tn_{2^n}^*]$. Thus the two solutions are related to each other in an even stronger sense than indicated by the above theorem.

V. Graph Families with Exact Bounds

The problem of precisely determining the standard chromatic number is an intractable one in general. Nonetheless, on specific families of graphs, such as the perfect graphs, it is possible to determine the chromatic number in polynomial time [Lov86]. But as we see in this section, even the simplest of the graph families that are so well-understood from the point of determining the chromatic number, such as the odd cycles, pose a rather difficult task if one is to determine their generalized chromatic number under arbitrary demand patterns. In what follows, we study various graph families on which we can both characterize and efficiently determine the exact generalized chromatic number. While these families are not broad enough to completely capture most of the widely used wireless network topologies, they do capture a variety of sub-topologies contained within them. As we see later, an understanding of these sub-topologies is valuable in determining both upper and lower bounds on the spectrum requirement.

A. Bipartite Graphs

We begin with a very simple class of graphs, namely the bipartite graphs. Let $G(V, E)$ be a bipartite graph; by definition, its vertices can be partitioned into subsets V_1, V_2 such that the edge set E contains only elements of the form (v_1, v_2) where $v_1 \in V_1$ and $v_2 \in V_2$. Given a demand vector $\mathbf{d}$, clearly, the least number of colors needed is bounded from below by the maximum demand, say N, along an edge. We show that there is a simple assignment scheme that can in fact match this lower bound. It is this nice property of bipartite graphs that results in the tractability of the *maximum packing algorithm* for dynamic channel assignment (this scheme requires repacking to admit each new call into the system) [EM82].

Our assignment scheme is as follows: if a vertex $v \in V_1$, assign it colors 1 through d_v, else assign it colors $(N - d_v + 1)$ through N. Since along an edge $(v_1, v_2) \in E$ we have $d_{v_1} + d_{v_2} \leq N$, it is readily seen that this coloring scheme is conflict-free. Thus we have

Theorem 2: For a bipartite graph G with an associated demand vector $\mathbf{d}$, $\chi^d(G) = \omega^d(G)$.

We point out that a *row of base-stations* with nearest neighbor interference is one of the simplest examples of a bipartite graph, and the above procedure does optimal allocation for arbitrary demand patterns on this graph. Another important example of this type of graph is the *regular square grid*, in which vertices indexed by (i, j) with $(i + j)$ odd have no edges to vertices with $(i + j)$ even. We note in the passing that a hexagonal grid topology under second tier interference assumption gives rise to graphs that are decomposable into four square grids.

B. Odd Cycles and their Complements

Bipartite graphs are examples of perfect graphs which by definition have $\chi = \omega$ (with unit demand at every vertex). The simplest examples of non-perfect graphs are odd cycles and their complements. For example, in case of odd cycles, the maximum clique size is $w(G) = 2$ while $\chi(G) = 3$. Note that even cycles are bipartite graphs. As we will see later in our analysis of the hexagonal grid topology, odd cycles play a central role in determining a lower bound on the spectrum requirements. Hence it is important to understand their behaviour under arbitrary demand vectors. Specifically, we would like to understand the behaviour of the ratio χ^d/ω^d under arbitrary demand vectors $\mathbf{d}$. By Proposition 1, we know that on an n-vertex odd cycle, $\chi^d \geq \max\{\omega^d, \gamma^d\}$ where $\gamma^d = \lceil \sum_i d_i / \lfloor n/2 \rfloor \rceil$. In what follows, surprisingly, we are able to show that $\chi^d \leq \max\{\omega^d, \gamma^d\}$ also holds, i.e., one of the two trivial lower bounds is always tight.

We now prove the above-mentioned result for odd cycles and also its analogue for the complement of odd cycles.

Lemma 1: For a given demand vector $\mathbf{d}$ on an odd cycle with n vertices, it is possible to assign one color along an independent set such that the remaining demand vector d' satisfies

$$\max\{\omega^{d'}, \gamma^{d'}\} = \max\{\omega^d, \gamma^d\} - 1$$

where $\gamma^d = \lceil \sum_i d_i / \lfloor n/2 \rfloor \rceil$. The same property holds for the complement of an odd cycle with γ^d defined as $\lceil \sum_i d_i / 2 \rceil$.

Proof: We begin with odd cycles. If one or more of the demands is zero, the graph is simply a collection of paths which are bipartite graphs, and hence satisfy $\chi^d = \omega^d \geq \gamma^d$. Thus in this case, there trivially exists an independent set that yields $\max\{\omega^{d'}, \gamma^{d'}\} = \max\{\omega^d, \gamma^d\} - 1$, since $\chi^{d'} = \chi^d - 1$. We assume from here on that all demands are non-zero.

Consider the number of cliques with maximum weight ω^d, say k^d. We must have

$$k^d \leq \lfloor 2 \frac{\sum d_i}{\omega^d} \rfloor$$

with $1 \leq k^d \leq n$. We then have the following cases.
1. $\omega^d \geq \gamma^d$. In this case, we must have $k^d \leq 2\lfloor n/2 \rfloor$, that is, at least one edge induces a clique $\mathcal{C}$ which is not of the maximum weight. We may then assign a color to the unique maximum independent set that does not hit $\mathcal{C}$ to reduce ω^d and γ^d by one.
2. $\omega^d < \gamma^d$. We must have $k^d > 2\lfloor n/2 \rfloor$ and hence $k^d = n$. For an odd cycle, this implies uniform demands. To see this, note that equal demands on all cliques, i.e. $d_1 + d_2 = d_2 + d_3 = ... = d_n + d_1$, implies $d_1 = d_2 = d_3 = ...d_n$ for n odd. Thus, assigning a color to any maximum independent set would reduce γ^d by one, while ω^d stays unchanged.
Similarly, for the complement of odd cycle, we have

$$k^d \leq \lfloor \lfloor n/2 \rfloor \sum_i d_i / 2 \rfloor$$

with the cases
1. $\omega^d \geq \gamma^d$, i.e. $k^d \leq 2\lfloor n/2 \rfloor$. There exists a unique independent set that hits every maximum clique, which is the pair of neighboring vertices in the corresponding odd cycle not visited by the non-maximum-weight clique $\mathcal{C}$.
2. $\omega^d < \gamma^d$. Again $k^d = n$ and all demands must be uniform by a similar argument. Assigning a color to any independent set reduces $\sum_i d_i/2$ by one.

■

Theorem 3: Given an odd cycle on n vertices with an associated demand vector $\mathbf{d}$,

$$\chi^d = max(\omega^d, \lceil \sum_i d_i / \lfloor n/2 \rfloor \rceil) \tag{4}$$

while for its complement,

$$\chi^d = max(\omega^d, \lceil \sum_i d_i / 2 \rceil) \tag{5}$$

Proof: Note that the theorem is a statement of the relative tractability of odd cycles and their complements, i.e. one of the two trivial bounds is always tight. We use induction, in conjunction with Lemma 1. Check that the bound is tight trivially for 0 demands. For non-zero demands, we may recursively find the independent set that reduces the bound by 1 and assign a color to it, thus decrementing the demand on this independent set by 1. This process may be continued till all demands vanish, by when we would have used exactly the stated number of colors. ∎

C. Graphs with Triangles as the only Odd Cycles

Here we consider an extension of bipartite graphs by allowing triangles to exist in the graph, but no odd cycles of length greater than 3. Bipartite graphs form the unique class of graphs with no odd cycles.

Theorem 4: Given a graph G that does not contain any odd cycle of size greater than three, and an associated demand vector $\mathbf{d}$, $\chi^d = \omega^d$.

Proof: We omit the proof due to space limitations. ∎

D. Perfect Graphs

Perfect graphs have several nice properties relating to coloring, as mentioned before. We now explore how these properties extend to generalized coloring. As one might expect, perfect graphs retain their tractability with regard to generalized coloring. However, as we shall see later, the hexagonal grid, even with interference restricted to the first tier, is not a perfect graph.

One method to prove the non-perfectness of a given graph is to show that an *induced* odd cycle, or its complement, can be embedded in the graph. In fact, the long unproven Strong Perfect Graph Conjecture (SPGC) states that a graph G is perfect if and only if it contains no induced odd cycles of size larger than three or their complements. No exceptions are known so far and the conjecture is widely held to be true. Note that bipartite graphs satisfy this criterion since they have no odd cycles, and hence no induced odd cycles either.

Prior to Lovasz's work [Lov72], it was believed that the notion of perfectness for unit demands was different from that for general demands. In this context, Fulkerson [Ful71], [Ful72] talks about *pluperfectness* for general demands and suggests that it may differ from perfectness, which would have disproved the SPGC following other known properties of perfect graphs. However, Lovasz subsequently showed that these two notions are identical. Lovasz proves the following property relating to perfect graphs.

Theorem 5 (Lovasz,1972) A graph G is perfect iff $\omega(H)\alpha(H) \geq |H|$ for all proper subgraphs H of G, including G itself.

To prove this theorem, Lovasz makes use of the property that replication of a vertex in a perfect graph does not destroy its perfectness, i.e.,

Lemma 2 (Lovasz,1972) Given a perfect graph G, one may construct other perfect graphs by replicating any vertex v, i.e. by adding a vertex v' such that v' has the same neighborhood as v. This property holds whether or not v' has an edge to v in the resulting graph.

We refer the reader to [Lov72] and references therein for the proofs of these results. However, we will use this result to restate the definition of perfectness as follows.

Theorem 6: A graph $G(V,E)$ is perfect iff $\omega^d(G)\alpha^d(G) \geq \sum_i d_i^2$ for any non-negative integer demand vector $\mathbf{d}$.

Proof: Observe that the vertex replication construction mentioned before does not destroy the perfectness of G, as stated by lemma 2. Each vertex with demand d_i is merely replaced by replicas without edges to the original vertex d_i times, followed by further replication of each replica d_i times with edges. In this expanded graph, the perfectness condition requires $\omega^d(G)\alpha^d(G) \geq \sum_i d_i^2$. However, since d_i can take any non-negative integer values, the set of demand vectors with binary entries produces all proper induced subgraphs of G. Since perfectness in the generalized context is identical to Lovasz's definition of perfectness, we may simply state that G is perfect iff this condition holds for arbitrary integer non-negative demand vectors $\mathbf{d}$. ∎

Conversely, $G(V,E)$ is non-perfect if $\omega^d(G)\alpha^d(G) < \sum_i d_i^2$ for some integer non-negative demand vector $\mathbf{d}$.

VI. Lower Bounds for Generalized Coloring

In this section, we investigate ways of lower-bounding the number of colors required for a given demand vector $\mathbf{d}$ on an arbitrary graph G with n vertices, maximum independent set size α, maximum weight clique size ω^d and maximum weight independent set size α^d.

As pointed out in Proposition 1, the most straightforward lower bounds are obtained as

$$\chi^d \geq max(w^d, \frac{\sum_i d_i}{\alpha})$$

since all colors in a maximum clique must be distinct and each color can be assigned to atmost α vertices. We showed before that this bound is tight for odd cycles. In general, however, as in the unit demand problem, these bounds can be considerably far from the chromatic number. Hence, we explore methods to obtain other non-trivial bounds for generalized coloring. This may be achieved using the vertex-replication construction. The following lemma establishes further correspondence between G and the expanded graph G'.

Lemma 3: G' has maximum clique size ω^d and maximum independent set size α^d coinciding with those of G.

Proof: If the maximum clique includes one of the vertices of any of the d_i cliques, it must include all vertices of that clique, since otherwise we may include the missing vertices to augment the clique. Similarly, the maximum independent set must include one vertex in each of the d_i cliques for a chosen vertex v_i, since otherwise a vertex from each of the the unrepresented cliques may be added to augment the independent set. Thus the set of vertices in G' replacing each vertex v_i in G contributes either none or exactly d_i vertices to the maximum clique and maximum

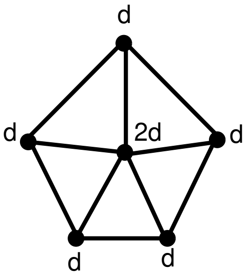

Fig. 1. The wheel with specified demands on the vertices.

independent set, which is its weight in G. Hence the maximum weight clique/independent set in G determines the vertices chosen in G' upto permutations within a replica set. ■

The number of vertices in G' is however much larger, $|G'| = \sum_i d_i^2 = \mathbf{d}^T\mathbf{d}$, and its chromatic number is lower bounded by $\chi^d \geq \frac{\sum_i d_i^2}{\alpha^d}$, which hence also lower bounds the number of colors needed to meet all demands in G. Thus, we have the following useful result:

Lemma 4: Given a graph G with an associated demand vector $\mathbf{d}$, $\chi^d \geq \frac{\sum_i d_i^2}{\alpha^d}$.

Thus $\chi^d \geq \max(\omega^d, \frac{\sum_i d_i}{\alpha}, \frac{\sum_i d_i^2}{\alpha^d})$. Another method to obtain this, and perhaps other tighter lower bounds in special cases, is through the dual of the LP formulation discussed in Section IV. We defer the details to a later version.

Example 1: To see the relevance of the bound of Lemma 4, consider the "wheel" graph shown in Figure 1, with a demand of $2d$ on the inner vertex and d on the outer vertices. In this case, $\omega^d = 4d, \sum_i d_i = 7d, \sum_i d_i^2 = 9d^2, \alpha = 2, \alpha^d = 2d$. The three lower bounds are hence given by

$$\begin{aligned}
\omega^d &= 4d \\
\frac{\sum_i d_i}{\alpha} &= 3.5d \\
\frac{\sum_i d_i^2}{\alpha^d} &= \chi^d = 4.5d
\end{aligned}$$

The last bound, obtained from the vertex-replication construction, is the tightest one, which happens to be exact in this case.

In general, the bound is slack for perfect graphs, since perfect graphs satisfy $\omega^d\alpha^d \geq \sum_i d_i^2$, which directly implies that $\omega^d \geq \sum_i d_i^2/\alpha^d$. However, for non-perfect graphs, there exist demand vectors for which $\omega^d < \sum_i d_i^2/\alpha^d$ and this bound is often better than the clique number.

Estimating α^d: The problem of determining α^d is NP-hard in general graphs. However, on graphs with low vertex degrees, for any fixed $\epsilon > 0$, it is possible to estimate α^d, in polynomial time, to within a $(1+\epsilon)$ factor of the optimal value [IMR+94]. Among other classes of graphs, these techniques apply to graphs that result from the hexagonal grid topology. The polynomial bounding the running time grows, however, as we choose smaller and smaller values for the ϵ.

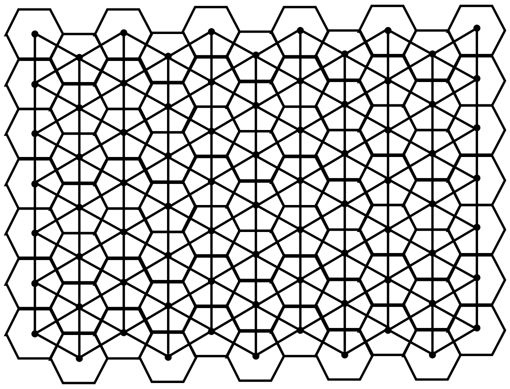

Fig. 2. Representation of a distance-1 graph.

VII. The Cellular Hexagonal Grid Graph

We now proceed to examine the hexagonal topology, which is usually the preferred layout of outdoor wireless networks, for specific interference structure and arbitrary demand patterns. Generally, the co-channel interference in cellular systems is a decaying function of the distance between the interfering cells. Hence, the interference graph for a standard hexagonal cellular pattern can be represented as a regular hexagonal grid with edges connecting neighbors within a fixed distance from any given node. Figure 2 depicts a hexagonal grid with edges between cells that are just a unit distance apart. We will analyze two primary cases of interest concerning hexagonal topology:

1. First, when only nearest neighbor interference is considered (first-tier interference), and
2. Second, when both nearest neighbor and next-to-nearest neighbor interference is considered (second-tier interference).

We begin with some definitions.

Definition 6 (Distance-1 Graph) A graph G is called a *distance-1 graph* if it can be embedded in a regular hexagonal grid such that there is an edge between any two vertices u and v in the graph iff the corresonding hexagon cells are adjacent to each other.

Definition 7 (Distance-2 Graph) A graph G is called a *distance-2 graph* if it can be embedded in a regular hexagonal grid such that there is an edge between any two vertices u and v in the graph iff the corresonding hexagon cells are either adjacent to each other or are both adjacent to a common cell (i.e are distance two apart).

Remark 1: In the subsequent discussion, we will assume that any distance-1 graph or distance-2 graph has an associated embedding in the hexagonal grid. In this embedding, each vertex of the graph is located at the center of its corresponding hexagonal cell and the edges correspond to the geometric straight-line connections between the embedded vertices. Our interest in the embedding stems from the fact that the geometry of the hexagonal layout plays an intrinsic role in the derivation of stronger bounds. The geometric techniques developed here may be extended to analyse distance-k graphs for larger k with some increased case analysis.

A. The First Tier Interference Graph

As pointed out earlier, a hexagonal grid with first-tier interference is shown in Figure 2. Note that the graph is planar, i.e., there are no intersecting edges in this embedding. While the graph satisfies $\chi = \omega = 3$; it is not a perfect graph, as can be seen from the embedding of an induced 9-cycle in Figure 6. Our main result for the first-tier intereference graphs is as follows:

Theorem 7: Given a distance-1 graph G with an associated demand vector $\mathbf{d}$,

$$\chi^d(G) \leq \frac{17}{12}\omega^d(G).$$

In contrast to above result, we later show that there exist instances of distance-1 graphs for which $\chi^d \geq 9\omega^d/8$ even when the demand is uniform at every node and tends to infinity asymptotically (allowing for a better re-use of colors).

In the next two subsections, we develop the theoretical foundations needed for establishing this bound. At the core of our proof lies a novel coloring scheme for distance-1 graphs that have no triangles. This coloring scheme is based on a decomposition strategy that strongly relies on the geometry of the distance-1 graph. The basic organization of our proof is as follows: When starting with an arbitrary distance-1 graph, we use an iterative 3-coloring scheme that satisfies the demand at sufficiently many nodes so that the remaining graph is triangle-free. Once we have transformed our starting graph to a triangle-free graph, we now use our new coloring scheme to efficiently color the remaining instance. We point out that using 3-coloring all along will not be nearly as efficient since for every iteration using three colors, the clique bound is at best guaranteed to go down by two units.

A.1 Triangle-Free Induced Subgraphs

We now examine induced subgraphs of the first tier interference graph that do not have any triangles embedded in them. Consider a distance-1 graph G that is triangle-free. Observe that even though G is triangle-free, its chromatic number may still be three; an example is when G is an odd cycle (see Figure 6). However, in this subsection, we establish that, given a uniform demand of $t > 1$ at every vertex, effectively $2.5t$ colors suffice to satisfy the demand in the entire graph. *While we observed the increased efficiency with increasing demands on isolated odd cycles before, it is striking to note that this property also holds for a triangle-free distance-1 graph that is an arbitrary collection of cycles. Perhaps, even more surprising is the fact that this improved efficiency in coloring surfaces with $t = 2$ itself.*

We begin by proving a technical lemma.

Lemma 5: Every vertex of a triangle-free distance-1 graph G can be covered twice using atmost five independent sets.

Proof: Since coloring assigned to two distinct components of a graph does not interfere, w.l.o.g. assume that G is connected.

We give an inductive proof, where we repeatedly remove sets of nodes from the graph, color the remaining graph inductively, and show how to extend the coloring to the removed set of nodes. Specifically, we show that G can be recursively decomposed into *chains* and *ears*. A chain is defined as a path $v_1, v_2, ..., v_k$ such that v_1 has degree one, every other v_i has degree two in G. An ear, on the other hand, is defined analogously but has the additional property that v_1 is also of degree two in G. Thus a chain is connected to the rest of the graph through only one of its end-points, while an ear has both its end points connected to the rest of the graph (possibly to the same vertex). In constructing our coloring, we label every vertex v of G using a pair of letters from the set $\{a, b, c, d, e\}$ denoting which among the five independent sets contains v.

If the graph G has a chain, remove it and color the resulting graph, say G', recursively. Let v' be the vertex in G' to which the chain was connected and let (a, b) denote the label assigned to v in the recursive coloring of G'. Then the coloring of G' can be extended in a conflict-free manner by assigning the label sequence $\{(c, d), (a, b), (c, d), ...\}$ to the vertices on the chain (starting at v).

Otherwise, every vertex of G must have degree at least two. We now argue that we can remove a suitable ear of size at least two from G and extend the coloring assigned to the resulting graph. To see this, first observe that since G is finite and has no vertices of degree less than two, its associated embedding must have a hull that is a proper, closed polygon. Let this polygon contain n sides. Since G is triangle-free, the only possible interior angles of the polygon are 120^o and 240^o, as can be seen by examining Figure 2. Note that the other possible interior angles, namely 60^o and 300^o, yield triangles at their locations. The sum of the interior angles of the polygon is given by

$$\sum_{i=1}^{n} \theta_i = (n-2)\,180^o = 120^o\, n_{120} + 240^o\, n_{240}$$

where n_{120} and n_{240} are the number of 120^o and 240^o interior angles respectively. Since $n = n_{120} + n_{240}$, it follows that

$$n_{120} - n_{240} = 6$$

which further implies that there is atleast one pair of 120^o angles occurring in succession. It is also easily seen that the smallest ear produced by a pair of successive 120^o angles is obtained when this pair occurs on successive vertices, which could result in an ear with 2 vertices as shown in Figure 3.

Note that, if either vertex x or y were of degree 2, we have an ear with more than 2 vertices. The smallest possible ear hence occurs when they both have degree 3 as shown. Thus G always has atleast one ear consisting of atleast 2 vertices. We remove the largest ear to get a graph G' and show that its coloring can be extended to assign each vertex of an ear to two independent sets. We first consider the case when the ear consists of 3 or more vertices.

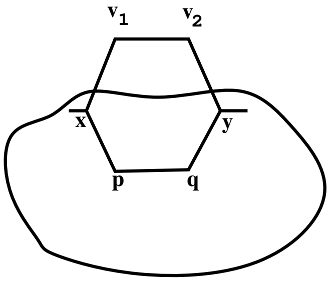

Fig. 3. An ear with 2 vertices v_1 and v_2; the cloud represents the remaining graph.

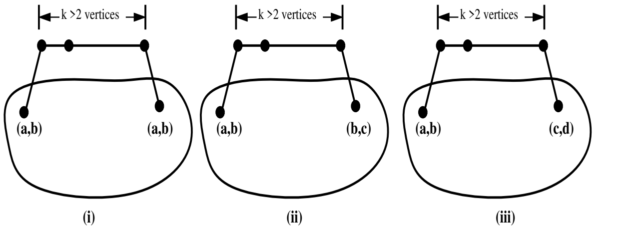

Fig. 4. Possibilities in the neighborhood of an ear with 3 or more vertices.

The possibilities are as shown in Figure 4, with (i) taking into account the possibility that endpoints may be attached to the same vertex. Each of the cases can be reduced to the problem of meeting demand 2 on every vertex of a cycle while fixing the assignment to 1 or more vertices. Figure 5 illustrates this idea. We first show that any arbitrary cycle with n vertices can be covered twice using atmost 5 independent sets. When n is even, 4 independent sets suffice, since 2 repeated uses of the independent sets containing odd and even numbered vertices covers all vertices twice. When n is odd, we use the independent sets $\{1,3,...,n-2\},\{2,4,....,n-1\},\{3,5,....,n\},\{4,6,....,n-2,1\},\{5,7,....,n-1,2\}$ to cover all vertices twice. Note that cases (i) and (iii) are straightforward from this argument, since they both involve covering the respective cycles as described. However, case (ii) is consistent with the covering specified above only for odd k. For even k, the sequence of pair-wise repeating assignments $(c,d),(a,e),(c,d),(a,e)...$, starting from the vertex immediately adjacent to the vertex marked (a,b), is conflict-free and covers all vertices twice. Thus, we have shown that any ear of size greater than 2 may be removed and replaced after the rest of the graph has been covered.

When the largest ear in G has only 2 vertices, the neighborhood looks as shown in 3. In this case, however, note that we may remove v_1 and v_2 and label the rest of G. When they are put back, v_1 is assigned the same labels as p and v_2 the same as q respectively with no conflicts. Thus, we have shown that every triangle-free induced subgraph of G can be twice-covered using atmost 5 independent sets. ■

This result may then be used to prove the following property of χ^d for triangle-free distance-1 graphs

Lemma 6: Given a triangle-free distance-1 graph G with arbitrary demands associated with every node, it can be colored with at most $5\omega^d/4+1$ colors.

Proof: We iteratively color the graph; in each iteration we use at most five colors. Let $G_0 = G$ denote the initial graph and let G_i denote the graph obtained after i iterations. Similarly, let ${\omega^d}_i$ denote the maximum clique demand in the graph G_i. An iteration involves the following steps:

1. For any isolated node v, satisfy $\min\{d_v, 5\}$ units of demand by assigning upto 5 colors.
2. Otherwise, in each non-trivial component of G_i, apply the scheme outlined in algorithm of Lemma 5. Using at most 5 colors, at every node v of the component, we satisfy $\min\{2, d_v\}$ units of demand. If in the current iteration, each demand is at least two, we are able to assign 5 colors such that each demand goes down by two. In particular, the maximum clique size goes down by 4. If any node has its demand completely satisfied, we delete it from the graph.

Let C be an arbitrary maximal clique in some component of G_i. We would now like to argue that essentially 4 units of demand are satisfied in C at end of the iteration. If the total demand of C is at least 4, and if C is either a single vertex clique or has both end-points with demand at least 2, we satisfy at least 4 units of demand in C.

The only difficult case is when C consists of a node u with demand exactly one connected to another node v. In this case, only 3 units of demand are satisfied in C. But observe that either v does not exist in G_{i+1} or it becomes an isolated vertex in G_{i+1}. And that in the very next iteration, it would have 5 units of demand satisfied. Thus ${\omega^d}_i \leq \omega^d - 4i + 1$.

In particular, substituting $i = \omega^d/4$, we get the desired result. ■

A.2 The General Case

We now build on the result of Lemm 6 to establish our main result, namely Theorem 7.

Proof of Theorem 7: The algorithm consists of two distinct phases:

1. In the first phase, we repeatedly 3-color the graph G so as to uniformly satisfy one unit of demand at every node — with the exception of isolated vertices; we satisfy 3 units

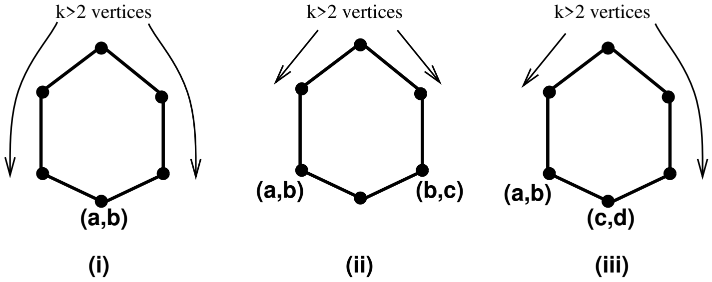

Fig. 5. Cycle representation of ears with 3 vertices or more.

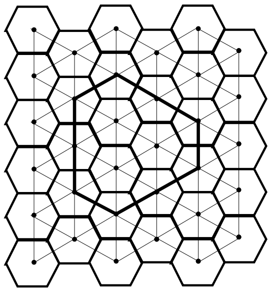

Fig. 6. An embedding of a 9-cycle.

of demand at every isolated vertex. Any vertex whose demand becomes zero, is deleted from the graph. This phase terminates when the resulting graph becomes triangle-free. Observe that the total number of iterations in this phase, say k, cannot exceed $\lfloor \omega^d/3 \rfloor$.

2. In the second phase, we apply the procedure outlined in the proof of Lemma 6. We observe that the max clique demand can be at most $\omega^d - 2k$. to satisfy the remaining demands by using at most $5(\omega^d - 2k)/4 + 1$ colors.

Thus the total number of colors used is bounded by

$$3k + \frac{5(\omega^d - 2k)}{4} + 1.$$

Using the fact that $k \leq \lfloor \omega^d/3 \rfloor$, we get the claimed result. □

A.3 Lower Bound

We now establish a lower bound on how far χ^d may deviate from ω^d. As shown before, odd cycles are graphs with $\chi^d > \omega^d$ in general, with the difference being more significant when the number of vertices is small. We hence seek to embed the smallest possible induced odd cycle in the distance-1 graph to estimate this difference, which can be seen, by inspection, to be a 9-cycle of the type shown in Figure 6. Consider a demand pattern with uniform demand d on this 9-cycle and zero demand everywhere else in the network. By Theorem 4, we have $\chi^d = 9d/4$, while $\omega^d = 2d$. Thus the $\chi^d/\omega^d = 9/8$ for this demand pattern, that is, the spectrum requirement may deviate by at least 12.5%, from the clique bound even in the simple case of only nearest-neighbor interference. It is worthwhile to note that this deviation also highlights the strength of the result of Lemma 6 since an induced cycle is triangle-free.

B. Frequency Assignment for the Second Tier Interference Graph

We now study the problem of frequency assignment when the interference from any neighbor within distance two is to be avoided. Perhaps, it is clear intuitively that the problem becomes more complex under this scenario. The following theorem of Simon [Sim88] provides a formal basis to this intuition.

Theorem 8: [Sim88] Given a distance-2 graph G with unit demand at every node, it is NP-complete to decide whether all demands can be satisfied by using atmost three colors.

That is, even determining the ordinary chromatic number of a distance-2 graph is NP-hard. In contrast, it is not difficult to verify that given any distance-1 graph G, its chromatic number can be determined in polynomial time. This readily follows from the fact that such an instance is always 3-colorable and if it requires less than three colors, then G must be bipartite and hence optimally colorable in polynomial time.

The intractability of determining $\chi(G)$ itself further manifests as difficulty in obtaining a strong general relation between ω^d and χ^d. Our main results are as follows:

Theorem 9: Given a distance-2 graph G with an associated demand vector $\mathbf{d}$, let $r = \omega^d/d_{min}$. Then

$$\chi^d(G) \leq \left(2 - \frac{1}{r}\right)\omega^d(G).$$

In contrast to the above result, we later show that there exist instances of distance-2 graphs for which $\chi^d \geq 5\omega^d/4$ even when the demand is uniform at every node and tends to infinity asymptotically (allowing for a better re-use of colors).

The bound stated in the above theorem is once again established via a geometric algorithm. The essential idea underlying above approach is to use a geometric argument to show that in any distance-2 graph, there exists a vertex v whose neighborhood has a strong structural property. We then use a recursive coloring approach whereby we first color the graph without the special vertex v and then argue using the structure of v's neighborhood that the demand at v can be satisfied via a small number of additional colors. The next subsection describes our algorithm and its analysis to establish Theorem 9.

B.1 The Upper Bound Results

We begin by establishing a lemma that is central to the algorithm we use in finding the desired coloring.

Lemma 7: Every non-trivial distance-2 graph G contains a vertex whose neighborhood can be partitioned into atmost two cliques.

Proof: If G contains a vertex of degree atmost one, we are done. Otherwise, the hull of G must be a proper closed polygon with, say, n sides. If θ_i denotes the i_{th} internal angle, then since $\sum \theta_i = (n-2)180^o$, there must exists a vertex v with internal angle at most 150^o. We argue that this vertex satisfies the desired property. Consider the neighborhood of a vertex in a 150^o cone as depicted in Figure 7 below.

The vertices labeled A through E form a clique and similarly, the vertices labeled F through I form another clique. Thus the neighborhood in a 150^o cone can be partitioned into atmost two cliques; the lemma follows. ■

We now state a simple coloring algorithm that uses the geometry of our interference graph; we refer to this algorithm as the `LEAST_ANGLE`.

`LEAST_ANGLE`

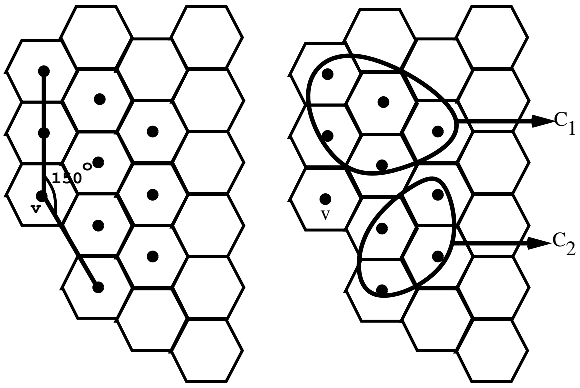

Fig. 7. The 150 cone

1. If G is not empty, find a vertex v with least internal angle in the hull of G; else return.
2. Delete v and recursively find a coloring of $G - \{v\}$.
3. Assign v the lowest available d_v colors that have not been assigned to any vertex in its neighborhood.

Proof of Theorem 9: We now show that algorithm `LEAST_ANGLE` colors any distance-2 graph G with an associated demand vector $\mathbf{d}$ in atmost $2\omega^d(G) - d_{min}$ colors, from which the theorem follows. By Lemma 7 we know that the vertex v removed in Step 1 of algorithm `LEAST_ANGLE` has the property that its neighborhood can be paritioned into two cliques $C_1(v)$ and $C_2(v)$. Let $D_1(v)$ and $D_2(v)$ respectively denote the total demand present at the vertices of $C_1(v)$ and $C_2(v)$. The largest number of colors used by the algorithm at any step is bounded by

$$\max_v \{d_v + C_1(v) + C_2(v)\}.$$

On the other hand, we know that

$$\omega^d \geq \max_v \{d_v + \max\{C_1(v), C_2(v)\}\}.$$

It follows immediately that

$$\begin{aligned} \max_v\{d_v + C_1(v) + C_2(v)\} \quad \leq \quad & 2 \max_v \{d_v + \\ & \max\{C_1(v), C_2(v)\}\} \\ & -d_{min}, \end{aligned}$$

which in turn is easily seen bounded by $2\omega^d - d_{min}$. □

B.2 Lower Bounds

We now establish a lower bound on how far χ^d may deviate from ω^d. As earlier, our lower bound proof is once again based on embedding an odd cycle. In this case, note that an induced 5-cycle can be embedded in the graph (see figure 8) thus we get $\chi^d = 5\omega^d/4$ when all the demand is distributed uniformly on a 5-cycle. This implies a 25% difference between clique estimates and chromatic number, which is substantial.

VIII. A Note on Adjacent Channel Interference

We recall another form of interference in cellular systems that plays an important role in channel assignment, and

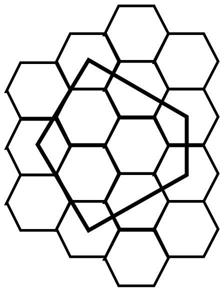

Fig. 8. Embedding of a 5-cycle in a distance-2 graph.

hence in spectrum estimation as well. *Adjacent channel interference*, which is usually weaker than the *co-channel interference*, arises from the fact that the frequencies in the spectrum have a natural ordering which rules out the use of nearby frequencies in a neighborhood. Here we consider only adjacent channel interference between frequencies assigned to the same base station. It is also possible for adjacent channel interference between adjacent base-stations to exist, but this effect is usually much weaker than the others.

It is easy to see that adjacent channel interference within a cell does not directly affect χ^d, which just denotes the number of distinct frequencies used in the network. However, it does affect another parameter of interest known as the *chromatic bandwidth*. For convenience, we recall the definition of chromatic bandwidth: Label the frequencies in the spectrum using positive integers $1, 2, \ldots.$ such that each frequency only interferes with other frequencies upto some distance m away in this ordering.

Definition 8 (Chromatic Bandwidth ψ^d) Given a graph G with an associated demand vector $\mathbf{d}$, and given a coloring $\mathcal{C}$ that assigns a set $\mathcal{C}_k$ of colors to vertex k, $1 \leq k \leq n(G)$ and $|\mathcal{C}_k| = d_k$, the chromatic bandwidth of the coloring $\mathcal{C}$ is defined as

$$\psi^d_{\mathcal{C}} = \max_k\{\mathcal{C}_k\} - \min_k\{\mathcal{C}_k\}$$

where $\max\{S\}$ ($\min\{S\}$) denotes the maximum (minimum) element of a set S. The chromatic bandwidth of graph G with an associated vector $\mathbf{d}$ is simply the minimum bandwidth over all colorings $\mathcal{C}$.

Assuming that frequency 1 is always used somewhere in the system, we have $\psi^d = \min_{\mathcal{C}} \max_k\{\mathcal{C}_k\}$.

Consider the case where only the adjacent frequencies interfere, i.e. i interferes only with $i-1$ and $i+1$. Note that, if all the demand were concentrated on a single vertex, $\psi^d = 2\chi^d$ ($2\chi^d - 1$ if χ^d is odd), whereas, on a bipartite graph with uniform demands on all vertices, $\psi^d = \chi^d$ since odd and even frequencies are assigned to different partitions. In general, we observe that $\psi^d \leq 2\chi^d$ since we may simply choose to use only odd numbered frequencies, for example. Thus we have the following simple proposition:

Proposition 3: Given a graph G with an associated demand vector $\mathbf{d}$,

$$\chi^d(G) \leq \psi^d(G) \leq 2\chi^d(G).$$

In the following, we discuss the effect of the above described form of adjacent channel interference on distance-1 and distance-2 graphs.

For the distance-1 graph, while $\psi^d = 2\chi^d$ in the worst case, one can an obtain an improvement if the graph is complete and no demand is very small. The basic idea is that the repeated use of 3-colorings of the graph to assign consecutive triples of frequencies is free of adjacent channel interference. Thus the total number of colors needed is at most $3d_{min} + 2(\omega^d - 2d_{min})$. Thus it is not difficult to conclude the following theorem:

Theorem 10: Given a distance-1 graph G with an associated demand vector $\mathbf{d}$, let $r = \omega^d/d_{min}$. Then

$$\psi^d = 2\omega^d(1 - \frac{1}{2r}).$$

We now describe a somewhat surprising result concerning adjacent channel interference on distance-2 graphs. We show that a modification of the algorithm `LEAST_ANGLE` can always color a distance-2 graph in $2\omega^d$ colors, respecting both the adjacent channel and the co-channel interference constraints.

Theorem 11: Given an instance G of second-tier assignment problem with demand vector $\mathbf{d}$, there exists an assignment that satisfies all the demands by using at most $2\omega^d$ colors with the property that no two adjacent channels are assigned to any single cell.

Proof: Let $\mathcal{C}$ denote the set of colors 1 through $2\omega^d$. The algorithm underlying the claimed bound is once again the algorithm `LEAST_ANGLE`. However, we need to slightly modify the algorithm. In the Step 3, when we color the vertex v, we assign it the colors as follows: let S be the set of colors that appear in v's neighborhood. By Lemma 7, since the neighborhood of v is partitionable into at most two cliques, we know that $|S| \leq 2(\omega^d - d_v)$. Thus either d_v odd numbered colors or d_v even-numbered colors have not been used in the assignment made to the neighborhood of v in the graph G'. We use the larger of the two sets to assign d_v colors to v. Clearly, no adjacent pair of frequencies gets assigned to v. ■

Remark 2: It is worthwhile to observe that, for arbitrary demand vectors on a graph G, the best possible guarantees on ψ^d are in the interval $[2\omega^d, 2\chi^d]$. Thus the result of Theorem 11 shows that the lower bound of $2\omega^d$ is always achievable on distance-2 graphs.

IX. Conclusions

We have studied the problem of frequency assignment as a generalization of the standard graph coloring problem. We established some useful properties of generalized coloring, and proved bounds on the generalized chromatic number of specific graph families. For the hexagonal grid topology, we developed specific geometric techniques to obtain strong bounds on the spectrum requirement. Our results concerning other specific graph families play a central role in revealing the structure that underlies the hexagonal grid. Though our focus has been on obtaining bounds, we also proposed efficient algorithms to perform allocations respecting the bounds.

Future work is aimed at further tightening the bounds, while developing and evaluating efficient algorithms for frequency assignment. In particular, we believe that our lower bound for the distance-2 hexagonal grid can be tightened, and there may exist demand patterns requiring upto $3\omega^d/2$ colors. We note in conclusion that in a real network, the spectrum requirement is strongly tied to the methods used for management of channels, and our bounds would be realistic only if the channel assignment algorithms used are efficient. Thus, the bounds may serve as benchmarks in assessing the absolute performance of static and dynamic channel assignment algorithms.

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