Edge Disjoint Paths Revisited

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Abstract

The approximability of the maximum edge disjoint paths problem (EDP) in directed graphs was seemingly settled by the $\Omega(m^{1/2-\varepsilon})$-hardness result of Guruswami et al. [10] and the $O(\sqrt{m})$ approximation achievable via both the natural LP relaxation [19] and the greedy algorithm [11, 12]. Here m is the number of edges in the graph. However, we observe that the hardness of approximation shown in [10] applies to *sparse* graphs and hence when expressed as a function of n, the number of vertices, only an $\Omega(n^{1/2-\varepsilon})$-hardness follows. On the other hand, the $O(\sqrt{m})$-approximation algorithms do not guarantee a *sub-linear* (in terms of n) approximation algorithm for *dense* graphs. We note that a similar gap exists in the known results on the integrality gap of the natural LP relaxation: an $\Omega(\sqrt{n})$ lower bound and an $O(\sqrt{m})$ upper bound. Motivated by this discrepancy in the upper and lower bounds we study algorithms for the EDP in directed and undirected graphs obtaining improved approximation ratios. We show that the greedy algorithm has an approximation ratio of $O(\min(n^{2/3}, \sqrt{m}))$ in undirected graphs and a ratio of $O(\min(n^{4/5}, \sqrt{m}))$ in directed graphs. For ayclic graphs we give an $O(\sqrt{n}\log n)$ approximation via LP rounding. These are the first sub-linear approximation ratios for EDP. Our results also extend to EDP with weights and to the unsplittable flow problem with uniform edge capacities.

1 Introduction

In this paper we consider the well-studied *edge-disjoint paths* problem (EDP). An instance of the EDP consists of a (directed or undirected) graph G and a set $\mathcal{T} = \{(s_i, t_i) : 1 \leq i \leq k\}$ of k source-sink pairs, and the objective is to connect a maximum number of these pairs via edge-disjoint paths. EDP is one of the classical NP-hard problems [7] and in addition to having several applications, most notably in VLSI routing, it is a problem of central importance in combinatorial optimization and graph theory. Consequently substantial effort has gone into understanding the approximability of this problem, several of its special cases, and generalizations. See [11, 12, 2, 10, 6, 1] and the references therein for the extensive literature.

Throughout the paper, general graphs refer to both directed and undirected instances, unless we explicitly indicate otherwise. The best known approximation ratio for general graphs is $O(\sqrt{m})$ [19, 12] where m is the number of edges in the graph. Essentially there have been two approaches (algorithms) to obtain upper bounds for general graphs: (randomized) rounding of the fractional solution to the natural multicommodity flow based relaxation [19, 2, 12, 16], and the greedy algorithm [11, 12, 10, 1]. We refer to the former as the LP approach and the latter as the greedy approach. Let $\mathcal{O}^*$ denote an optimal solution to the LP relaxation for a given instance and let OPT denote the integral optimum value for the same instance. It is clear that $\mathcal{O}^* \geq$ OPT. The best known results for EDP are summarized in [12]. For the LP approach, Srinivasan [19] shows that randomized rounding obtains $\max\{\frac{\mathcal{O}^*}{\sqrt{m}}, \frac{(\mathcal{O}^*)^2}{m}, \frac{\mathcal{O}^*}{d}\}$ paths where d is average length of flow paths in the LP solution. Kolliopoulos and Stein [12] show similar bounds for the greedy algorithm obtaining $\max\{\frac{\text{OPT}}{\sqrt{m_0}}, \frac{\text{OPT}^2}{m_0}, \frac{\text{OPT}}{d_0}\}$ paths where m_0 and d_0 are the number of edges and average length of paths respectively, in some optimum solution.

EDP in directed graphs appears to be fundamentally harder than in undirected graphs. For example, given a directed graph G, the problem of deciding if two given pairs, (s_1, t_1) and (s_2, t_2), can be connected via disjoint paths is already NP-hard [5]. In undirected graphs, for any fixed k, Robertson and Seymour [17] gave a polynomial time algorithm to decide if a given set of k pairs can be connected via disjoint paths. However, their algorithm is highly non-trivial and requires extensive machinery from the work on graph minors. Surprisingly, there is a fairly simple algorithm for the same problem in directed *acyclic* graphs [5]. Until recently, it was only known that EDP is APX-hard [9]. Guruswami et al. [10], via a simple reduction to the two pair decision problem mentioned above, showed that it is NP-hard to approximate EDP in directed graphs to within a $\Omega(m^{1/2-\varepsilon})$ factor for every fixed $\varepsilon > 0$. Independently, Ma and Wang [15] showed, via a more complex and indirect reduction from the maximization version of the label cover problem, that EDP in directed graphs is $\Omega(2^{\log^{1-\varepsilon} n})$-hard. Although the hardness result in [10] is stronger, we observe that the instances generated by the construction in [15] are *acyclic* while the reduction in [10] uses graphs with cycles

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in an essential way.

At first glance it would appear that the complexity of the directed EDP has been resolved. However, this is only true if we use m as the parameter for expressing the approximability of EDP. The hardness of approximation shown in [10] applies only to *sparse* graphs. When expressed as a function of n, the number of vertices in the graph, the hardness of approximation obtained is $\Omega(n^{1/2-\varepsilon})$. An obvious question to ask is if one can obtain an $o(n)$-approximation for EDP in directed graphs. In particular, is there an $O(\sqrt{n})$ approximation algorithm? A positive answer to this question would settle the approximability of the problem, and would simultaneously improve the current ratio of $O(\sqrt{m})$ for all graphs with super-linear number of edges, and by a substantial factor for *dense* graphs. Furthermore, answering this question would help us understand whether m or n is the right parameter to express the approximability results for EDP. We note that the *vertex-disjoint paths* problem is a special case of the EDP in directed graphs for which an essentially tight $\Theta(\sqrt{n})$ approximability threshold has been established [10].

In this paper, we make progress towards answering the above question in the positive by obtaining an $O(n^{4/5})$ approximation for EDP in directed graphs. For directed *acyclic* graphs we obtain an $O(\sqrt{n}\log n)$-approximation. For undirected graphs we obtain an $O(n^{2/3})$-approximation. These are the first sub-linear (in n) approximation guarantees for the problem.

Overview of Results and Techniques: Our first set of results are based on the analysis of the greedy algorithm for EDP [12] that iteratively picks a shortest path among unrouted pairs [1]. We show that greedy is an $O(\min\{n^{2/3}, \sqrt{m}\})$-approximation algorithm on undirected graphs and an $O(\min\{n^{4/5}, \sqrt{m}\})$-approximation algorithm on directed graphs. An upper bound of $O(\sqrt{m})$ on the approximation ratio of greedy is known [12] and is based on two observations: if a greedily chosen path is "short" (has $O(\sqrt{m})$ edges) then it can result in the removal of only $O(\sqrt{m})$ paths in some fixed optimum solution; if it is "long" (has $\Omega(\sqrt{m})$ edges), then the graph has capacity to route only $O(\sqrt{m})$ pairs in the residual graph. The key observation underlying our analysis of the greedy algorithm is that when there are no short paths, the capacity of the graph to route additional pairs is more strongly diminished than suggested by the above counting argument. Even and Tarjan [4] showed that in a simple unit capacity directed graph, for any two vertices s and t, the maximum flow from s to t is $O(n^2/\ell^2)$ where ℓ is the shortest path distance from s to t. Our analysis of the greedy algorithm is based on the following question. Given a unit capacity graph G, let S_ℓ be the set of all ordered pairs (s_i, t_i) such that $\mathrm{dist}_G(s_i, t_i) \geq \ell$. What is the maximum *multi-commodity* flow[2] between pairs in S_ℓ? We show via a simple counting argument that in undirected graphs the maximum flow is $O(n^2/\ell^2)$. However, the argument does not generalize to directed graphs. We use a different idea based on the recent work of Cheriyan, Karloff, and Rabani [3] on directed multicuts to show that there exists a cut of value $O(n^4/\ell^4)$ that separates all pairs in S_ℓ. This of course also implies that the maximum flow is $O(n^4/\ell^4)$.

The bounds we establish on the greedy algorithm can be translated into similar upper bounds on integrality gap of the natural LP formulation. We are thus able to improve the gap between the known upper bound of $O(\sqrt{m})$ [19] and lower bound of $\Omega(\sqrt{n})$ [9]. We show that the greedy algorithm has an improved approximation in undirected graphs if the graph is uniformly dense. In particular we show that in graphs with minimum degree δ, greedy yields an $O(n/\delta)$ approximation. Hence if δ is $\Omega(n^{1-\varepsilon})$ we obtain an $O(n^\varepsilon)$ approximation. Our analysis for this case uses a charging scheme that is different from the one for the general case. We also show that there are an infinite family of undirected (and directed) graphs on which the approximation ratio obtained by greedy is $\Omega(n^{2/3})$, thus our analysis of greedy is tight for undirected graphs.

Our second result is for EDP in directed *acyclic* graphs. For these graphs we obtain an $O(\sqrt{n}\log n)$ approximation. We also establish the same upper bound on the LP integrality gap thus matching to within a log factor the $\Omega(\sqrt{n})$ lower bound in [9] that applies to acyclic graphs as well. Recall that in [15] an $\Omega(2^{\log^{1-\varepsilon} n})$-hardness is shown for acyclic graphs. We note that our $\Omega(n^{2/3})$ lower bound on the approximation ratio of greedy applies to acyclic graphs, hence we show that the LP approach is stronger for a natural class of directed graphs. We also show that if congestion 2 paths are allowed then we can obtain the same $O(\sqrt{n}\log n)$ bound for general graphs. With congestion 2 paths an $O(\sqrt{n})$ bound is already known [14], however we obtain this result as a byproduct of our algorithm for acyclic graphs. The underlying idea behind both algorithms is novel and we describe it next.

The difficult case for all existing approaches to the EDP is when the paths chosen by the optimal algorithm are $\Omega(\sqrt{n})$ long. We address this case as follows: a simple counting argument shows that if there are many such paths, then there must a vertex v with high congestion (that is many paths go through it). In order to take advantage of this observation we introduce a restricted class of instances that we call v-*separable* instances. Informally speaking, these instances have the property whereby the vertex set can be partitioned into two sets — one containing only sources and

[1] Kleinberg [11] defined an online *bounded* greedy algorithm that picks a pair as long as the shortest path between them is at most a parameter L. Our analysis is in fact for the bounded greedy algorithm, however we use greedy algorithm as defined in [12] since it is a more natural offline algorithm.

[2] There are two variants of multi-commodity flow: maximum concurrent flow, and maximum sum multi-commodity flow. In this paper we are concerned only with sum multi-commodity flow.

other only sinks — such that v is the only common vertex in these two sets, hence all paths between the pairs share the bottleneck vertex v. Such instances naturally define two single source disjoint paths problems. We use an LP-based approach to obtain an $O(\log^2 n)$-approximation for v-*separable* instances. In acyclic graphs, every vertex v naturally defines a v-separable instance, and this is the key to obtaining our $\tilde{O}(\sqrt{n})$ approximation. In general graphs, a similar idea works but we get congestion 2 paths. We observe that although we obtain paths that intersect, they do so in a very restricted fashion and we believe that they can be uncrossed to obtain improved ratios for undirected graphs. Based on our results we conjecture the following.

Conjecture *There is an $O(\sqrt{n})$-approximation for EDP. Moreover, the LP integrality gap is $\Theta(\sqrt{n})$.*

Extensions: Although we concentrate on the basic edge disjoint paths problem in this paper, our ideas extend to some generalizations of the problem. In particular, with a loss of additional logarithmic factor in the approximation ratio, it is straightforward to extend our results to the *weighted EDP*: each pair (s_i, t_i) has a weight associated with it and the goal is to find a maximum weight set of pairs that can be routed. We also show that all our bounds hold for the uniform capacity *unsplittable flow problem*: each pair (s_i, t_i) has a demand associated with it and all edges have identical capacity.

Organization: We start by describing the multicommodity flow based LP relaxation for EDP in Section 2. In Section 3, we present a new analysis of the greedy algorithm as well as our upper bound on the LP integrality gap. We also show here that the greedy algorithm can be used to get stronger approximation guarantees when the underlying graph has a large minimum degree. In Section 4 we describe our $O(\sqrt{n}\log n)$ approximation for directed acyclic graphs. To obtain this result we formalize the notion of a v-separable instance and show that a good approximation for v-separable instances translates into a good approximation for EDP in directed acyclic graphs. In Section 5 we give an LP-based approximation algorithm for v-separable instances. Section 6 describes extension to the uniform capacity unsplittable flow problem. Finally, Section 7 has concluding remarks in which we explore how our ideas from Sections 4 and 5 could potentially extend to the general case.

2 Preliminaries

EDP has a natural IP formulation based on multicommodity flow. Here we give the path formulation for convenience, there is an equivalent and compact flow formulation.

Let $\mathcal{P}_i$ denote the set of all paths in G from s_i to t_i. For each path π in $\cup_i \mathcal{P}_i$ we have a binary variable f_π that indicates if π is chosen in the solution. For each pair (s_i, t_i) we have a variable x_i indicating whether the pair is chosen to be connected. By relaxing the final set of constraints to $0 \le f_\pi \le 1$ for $\pi \in \bigcup_{i=1}^k \mathcal{P}_i$ we obtain an LP relaxation that we shall refer to as LP-MCF . Given a fractional solution $(x_1, \dots, x_k, f_{\pi_1}, f_{\pi_2}, \dots)$, we shall refer to $\sum_{i=1}^k x_i$ as the value of the solution. We shall say that the solution uses a flow path π if $f_\pi > 0$. An optimal solution to LP-MCF can be obtained in polynomial time using standard techniques. Simple and efficient combinatorial methods that obtain a $(1+\varepsilon)$ approximation are also known [8].

$$
\begin{aligned}
&\max \sum_{i=1}^k x_i, \text{ s.t.} \\
&\sum_{\pi \in \mathcal{P}_i} f_\pi = x_i, && i = 1, \dots, k \\
&\sum_{i=1}^k \sum_{\pi \in \mathcal{P}_i : \pi \ni e} f_\pi \le 1, && e \in E(G) \\
&x_i \le 1, && i = 1, \dots, k \\
&f_\pi \in \{0, 1\}, && \pi \in \bigcup_{i=1}^k \mathcal{P}_i.
\end{aligned}
$$

3 The Greedy Algorithm

We show that the well-studied greedy algorithm gives an approximation ratio of $O(n^{2/3})$ for the undirected disjoint paths problem and an approximation ratio of $O(n^{4/5})$ for the directed case. The algorithm simply considers the shortest path length for each unrouted (s_i, t_i) pair and connects a pair with minimum shortest path length via its shortest path. The earlier analysis of the greedy algorithm [12] used the fact that when the minimum path length for unrouted pairs exceeds a certain bound, say ℓ, then OPT can route only $O(m/\ell)$ pairs. We show that when m is large, the capacity of the network to route pairs in this case is more restricted than suggested by this bound. In particular, we show that in the undirected case, an optimal solution can route at most $O(n^2/\ell^2)$ pairs while in the directed case, it can route at most $O(n^4/\ell^4)$ pairs. These bounds apply even to the fractional optimum and as a result, allow us to obtain the first sub-linear bounds on integrality gaps for the natural LP formulation described in the preceding section. We start by establishing these bounds.

THEOREM 3.1. *Let $G(V, E)$ be a simple unit capacity undirected graph and let $\mathcal{T}$ be the collection of all source-sink pairs such that the shortest distance in G between source and sink is at least ℓ. Then the value of the maximum (fractional) multicommodity flow for the pairs in $\mathcal{T}$ is bounded by $O(n^2/\ell^2)$.*

Proof. Let us call a vertex in G to be a high degree vertex if its degree is strictly greater than $6n/\ell$ and low degree otherwise. We claim that for every pair (s, t) in $\mathcal{T}$, any path from s to t must use at least $\ell/6$ low degree vertices. It then follows that a fractional flow path carrying a flow of $\varepsilon > 0$ consumes $\Omega(\varepsilon \cdot \ell)$ capacity incident on the low-degree vertices. Thus total fractional flow that can be supported by G is $O(n^2/\ell^2)$ since total capacity incident on the low degree vertices is bounded by $O(n^2/\ell)$.

We prove the above claim by contradiction. Suppose there exists a path P that connects a pair $(s,t) \in \mathcal{T}$ using less than $\ell/6$ low degree vertices. Consider a breadth-first search tree rooted at s, and let *layer* L_i denote the set of vertices at distance i from s. It is easy to see that every edge connects vertices that are either in the same layer or in two adjacent layers. We claim that at least $\ell/6$ layers among the first ℓ layers must consist only of low degree vertices. Suppose not. Partition the layers into blocks of 3 contiguous layers each such that the block B_i consists of vertices in the layers L_{3i+1}, L_{3i+2}, and L_{3i+3}. Discard any blocks that contain a layer consisting only of low degree vertices. By our assumption on P, at most $\ell/6$ blocks are discarded and thus we remain with at least $\ell/6$ blocks. Since the blocks are disjoint, one of the remaining blocks contains at most $6n/\ell$ vertices altogether. Consider the high degree vertex in the middle layer of such a block — all its edges go to other vertices within the block. A contradiction! Thus there must be at least $\ell/6$ layers among $L_1, L_2, ..., L_\ell$ that consist entirely of low degree vertices. Now any s-t path must cross these layers and thus hit at least one low degree vertex in each such layer. The claim, and hence the theorem follows. $\square$

The preceding argument does not extend to the directed case. The technical difficulty is that in a breadth first tree from a given vertex in a directed graph, even though there cannot be forward arcs between non-adjacent layers, there could be many back arcs between non-adjacent layers. It is easy to construct graphs where the in-degree and the out-degree of every vertex is $\Omega(n)$ but there still exists an s-t pair such that the shortest path distance from s to t is $\Omega(n)$.

For the directed case we use a different argument that is in part inspired by the recent algorithm and analysis of Cheriyan, Karloff and Rabani [3] for the directed multicut problem. They find a cut based on an LP relaxation while we find a cut using the fact that pairs are far apart in the graph.

THEOREM 3.2. *Let $G(V,E)$ be a simple unit capacity directed graph and let $\mathcal{T}$ be the collection of all source-sink pairs such that the shortest distance in G between source and sink is at least ℓ. Then there exists a cut of size $O(n^4/\ell^4)$ that separates all pairs in $\mathcal{T}$, and hence the value of the maximum (fractional) multicommodity flow for the pairs in $\mathcal{T}$ is bounded by $O(n^4/\ell^4)$.*

Proof. In the following, pairs are ordered and when we say an s-t path we mean a directed path from s to t. Separating s from t does not imply that t is separated from s.

We show that there exists a set $E' \subseteq E$ of size $O(n^4/\ell^4)$ such that by deleting edges in E', we can separate every source-sink pair in $\mathcal{T}$. We construct the set E' iteratively as follows. Initialize E' to $\emptyset$. We denote the residual graph $G(V, E \setminus E')$ by G'. At the beginning of the iteration, we pick any pair $(s,t) \in \mathcal{T}$ such that G' contains an s-t path. We do a breadth-first search from s in the graph G'. Let layer L_j denote the vertices at distance j from s. We partition these layers into blocks of 2 adjacent layers each such that the ith block B_i consists of vertices in the layers L_{2i+1} and L_{2i+2}. By our assumption, t occurs in some block B_r where $r \geq \ell/2$. Let $Y = B_1 \cup B_2 ... \cup B_{\ell/6}$ and let $Z = B_{\ell/3+1} \cup B_{\ell/3+2} ... \cup B_r$. Consider the $\ell/6$ blocks not contained in Y or Z, namely, the blocks $B_{\ell/6+1}$ through blocks $B_{\ell/3}$. Since the blocks are disjoint, there exists a block B_k among these blocks such that the total number of vertices in B_k does not exceed $6n/\ell$. We include in E' all edges within this block, that is, edges that go from layer L_{2k+1} to L_{2k+2} — the total number of these edges is $O(n^2/\ell^2)$. After we remove these edges it is clear that there is no s-t path in G'. We iterate until there are no pairs in $\mathcal{T}$ left to separate. Now we argue that the total number of iterations cannot exceed $O(n^2/\ell^2)$. This combined with the fact that we remove $O(n^2/\ell^2)$ edges in each iteration gives us the desired bound on E'.

Consider an iteration above that separates the pair s and t and let P be a shortest path from s to t in G'. From the above arguments it clear that not only do we separate s from t, we also separate Y from Z. In particular we separate $P \cap Y$ from $P \cap Z$. Note that since P is a path, every vertex in $P \cap Y$ could reach every vertex in $P \cap Z$ before this iteration. The sets $P \cap Y$ and $P \cap Z$ contain at least $\ell/3$ vertices each, thus each iteration separates $\Omega(\ell^2)$ pairs of vertices that were connected before the iteration. Hence the total number of iterations is $O(n^2/\ell^2)$. $\square$

We now show how the above theorems can be used to obtain tighter bounds on the performance of the greedy algorithm. For the purpose of our analysis, we will find it more convenient to work with the variation of this algorithm that terminates as soon as the minimum shortest path length among unrouted pairs exceeds a certain bound, say ℓ. Clearly, the unrestricted greedy algorithm dominates the performance of this restricted version. The analysis below applies to undirected as well as directed graphs. Unless necessary, we will not make any explicit reference to the directed or the undirected nature of the underlying graph.

Let S be the set of paths chosen by the greedy algorithm. Fix an optimal solution, say OPT. Let O_u denote the set of pairs that are routed by OPT and not by the greedy algorithm. If a path in O_u shares an edge with a path $P \in S$, we charge it to the path P in the greedy solution. Let O_1

denote the set of paths in O_u that can be accounted for in this manner. Clearly, $|O_1| \le |S| * \ell$. Let $O_2 \stackrel{\text{def}}{=} O_u \setminus O_1$. Also, let $G_r(V, E_r)$ be the subgraph of G that remains at the termination of the greedy algorithm. Applying Theorems 3.1 and 3.2 to the graph G_r, it follows that $|O_2|$ is $O(n^2/\ell^2)$ in the undirected case and $O(n^4/\ell^4)$ in the directed case. Thus choosing a threshold of $\ell = n^{2/3}$ for the undirected case, and a threshold of $\ell = n^{4/5}$ in the directed case gives us the following result.

THEOREM 3.3. *For EDP, the greedy algorithm gives an $O(n^{2/3})$-approximation in undirected graphs and an $O(n^{4/5})$-approximation in directed graphs.*

3.1 Lower Bound for Greedy We next show that the performance ratio of the greedy is at least $\Omega(n^{2/3})$ for both undirected and directed EDP. Thus our analysis is asymptotically tight in at least the undirected setting.

THEOREM 3.4. *There exists an infinite family of directed and undirected instances on which the approximation ratio achieved by the greedy algorithm is $\Omega(n^{2/3})$. Moreover, the directed instances in the family are acyclic.*

Proof. We now describe a family of instances on which the greedy algorithm may end up routing only one path while there always exists a solution of size $\Omega(n^{2/3})$. We focus on the directed case here; however, the underlying undirected graph serves as a lower bound example for the undirected case. The instance consists of two layered graphs, say G_1 and G_2, where each G_i contains $n^{2/3}$ layers of $n^{1/3}$ vertices each. For notational ease, let $p = n^{1/3}$ and let $q = n^{2/3}$. Let $X_1, ..., X_q$ be the layers of G_1, and let $Y_1, ..., Y_q$ be the layers of G_2. Then for $1 \le i < q$, every vertex in X_i is connected to every vertex in X_{i+1}. Y_i is connected to Y_{i+1} in a similar way. The two graphs are connected as follows. For each odd i, we pick a representative vertex in X_i, say x_i, and for each even i, we pick a representative vertex in Y_i, say y_i. We then add edges (x_1, y_2), (y_2, x_3),...,(x_{q-1}, y_q) — assume without any loss of generality that q is even. The source sink pairs are given by the set $\{(x, y) \mid x \in X_1, y \in Y_q\}$. This completes the description of our instance. See Figure 1.

Observe that every source-sink path is of length at least $q-1$ in this instance. Suppose the greedy algorithm chooses to connect the pair (x_1, y_q) along the path $x_1 \to y_2 \to ...x_{q-1} \to y_q$. Note that this path is a shortest x_1-y_q path. (By a simple modification to the graph we can ensure that this path is the unique shortest path between unrouted pairs.) Then all remaining source-sink pairs are disconnected in the residual graph and thus no more pairs can be routed by the greedy. On the other hand, we claim that an optimal algorithm can route $\Omega(n^{2/3})$ pairs. Notice that by our choice of the source-sink pairs, it suffices to show that any cut that separates X_1 from Y_q must have capacity at least $\Omega(n^{2/3})$. In

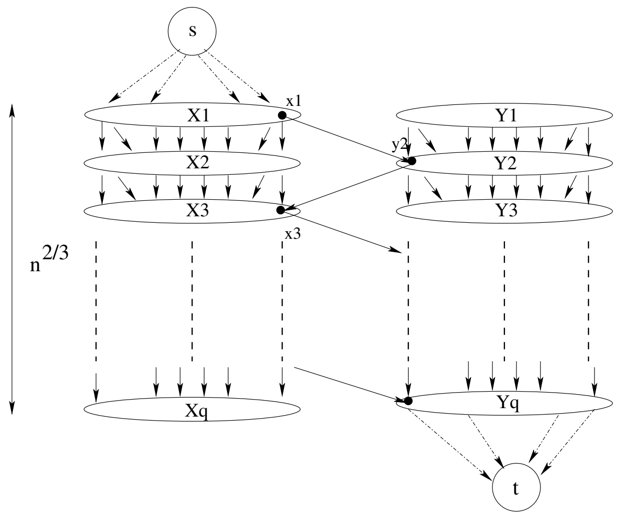

Figure 1: A lower bound instance for the greedy algorithm. (Vertices s and t are only for the proof).

order to show this, it is convenient to imagine a super-source vertex s connected to X_1 via edges of infinite capacity, and a super-sink vertex t to which all vertices in Y_q are connected by edges of infinite capacity. We will argue that any s-t cut has capacity at least $n^{2/3}/4$. Consider any such cut (S, T). Suppose there exists a layer X_i such that $|S \cap X_i| \le n^{1/3}/2$. Consider the smallest index i for which this occurs. Since $|S \cap X_{i-1}| \ge n^{1/3}/2$, it is easy to see that the capacity of the (S, T) cut is at least $n^{2/3}/4$ and we are done. Thus we may assume that $|S \cap X_i| \ge n^{1/3}/2$ for each X_i. A similar argument implies that $|T \cap Y_j| \ge n^{1/3}/2$ for each Y_j. Now suppose S contains more than $n^{1/3}/2$ representative vertices from G_2. Then by the preceding condition, each such vertex has at least $n^{1/3}/2$ neighbors in T and thus the capacity of the cut is at least $n^{2/3}/4$. So it must be that at least $(n^{2/3} - n^{1/3})/2$ of the representative vertices in G_2 are in T. A similar argument shows that at least $(n^{2/3} - n^{1/3})/2$ of the representative vertices in G_1 are in S. This once again implies that the capacity of the cut is at least $n^{2/3}/4$. □

3.2 Integrality Gap of the LP Formulation The preceding analysis of the greedy algorithm can be used to obtain improved upper bounds on the integrality gap of LP-MCF in directed and undirected graphs. The best previous bound is $O(\sqrt{m})$. A lower bound of $\Omega(\sqrt{n})$ is known even in directed acyclic graphs and it remains open if a matching upper bound exists. As we will shortly see in Section 4, this question can be answered in the affirmative for directed acyclic graphs.

THEOREM 3.5. *The integrality gap of* LP-MCF *in undirected graphs is $O(n^{2/3})$ and in directed graphs is $O(n^{4/5})$.*

Proof. Let $\ell = n^{2/3}$ in the undirected case and $\ell = n^{4/5}$

in the directed case. Given a feasible fractional solution, let us consider the subgraph G' of G that contains edges with positive fractional flow. We now run the restricted greedy algorithm on G' (i.e. terminate once the shortest path length for each unrouted pair exceeds ℓ). Let S be the set of paths chosen by the greedy. We will focus only on pairs (fractionally) routed by the fractional solution but unrouted by the greedy. Fix a decomposition of the fractional solution into fractional flow paths, and discard any flow paths for pairs that are routed by the greedy algorithm. Any fractional flow path that intersects with a path in S is charged to S. Clearly, total flow on the flow paths charged in this manner is bounded by $|S| \cdot \ell$.

Now consider the residual graph G_r that consists of edges in G' that do not intersect with S. Any unaccounted fractional flow path is strictly contained in G_r. As before, the total fraction flow that can be routed in G_r can be bounded using Theorems 3.1 and 3.2. Putting together, we get the claimed bounds. $\square$

We note here that the algorithm used in Theorem 3.5 ignores the flow paths of the fractional solution. This is in contrast to randomized rounding techniques that route each pair on some flow path for that pair provided by the fractional solution.

3.3 Undirected Dense Graphs: We show that in dense graphs (i.e. graphs where the minimum degree is large), the greedy algorithm gives much improved approximation ratios. Our analysis of this case differs from that of the general case. Specifically, we show that any pair that is routed by an optimal solution and is not routed by the greedy, must pass through a "heavily-used" vertex in the greedy solution. As a result, we can account for these unrouted pairs by charging to the paths in the greedy algorithm that pass through these heavily-used vertices.

THEOREM 3.6. *The approximation ratio of the greedy algorithm is $O(n/\delta)$ on undirected graphs with minimum degree at least δ. In particular, for any $0 \le \varepsilon < 1$, on graphs with minimum degree $\Omega(n^{1-\varepsilon})$, the approximation ratio of greedy is $O(n^{\varepsilon})$.*

Proof. As before, we work with a restricted version of greedy that stops when the shortest path length exceeds a certain bound — we use a bound of $6n/\delta$. Upon termination of the restricted greedy, let G_r denote the residual graph. We partition the vertices into two sets, namely, *saturated* and *unsaturated*, based on the ratio of their degrees in G and G_r. A vertex is called saturated if its degree G_r is less than half of its initial degree in G and unsaturated otherwise.

Let S be the set of paths chosen by the greedy algorithm. Fix an optimal solution, say OPT, and let O_u denote the set of pairs in OPT not connected by the greedy. For each saturated vertex v, define S_v to be the subset of paths in S that pass through vertex v. If a path in O_u passes through a saturated vertex v, we simply charge this path to the set S_v. Clearly, since v is saturated, at most $2|S_v|$ paths in O_u are charged to a set S_v in this manner. On the other hand, $\sum_v |S_v| \le (6n/\delta)|S|$. Thus we account for at most $12(n/\delta)|S|$ paths in O_u in this manner; let O_1 denote this set. We now consider $O_2 \stackrel{\text{def}}{=} O_u \setminus O_1$. Every path in O_2 passes through only vertices that are unsaturated and hence have degree at least $\delta/2$ in G_r. An elementary result in graph theory shows that any connected graph with minimum degree δ_0 has diameter at most $3n/\delta_0$ [20]. Thus any pair in O_2 can be connected by a path of length at most $6n/\delta$. This contradicts the fact that the greedy algorithm terminated. Thus $O_2 = \emptyset$. Putting together, $|\text{OPT}| \le |S| + 12(n/\delta)|S|$ and thus the theorem follows. $\square$

The above theorem does not hold in directed graphs for reasons similar to those mentioned before Theorem 3.2.

4 An $\tilde{O}(\sqrt{n})$ approximation for acyclic graphs

In this section we obtain an $O(\sqrt{n}\log n)$-approximation algorithm for the edge disjoint paths problem on acyclic graphs, based on the LP relaxation. We are thus able to match to within a log factor the known lower bound of $\Omega(\sqrt{n})$ on the integrality gap. For a large class of general graphs our result is the first to provably demonstrate that the LP provides asymptotically better approximation ratios than the greedy algorithm. We also remark that in contrast to randomized rounding approaches our algorithm routes pairs on paths that are not necessarily provided by the LP solution.

The Main Idea: We now describe the key high level idea behind our algorithm. To obtain an $\tilde{O}(\sqrt{n})$ approximation algorithm it is sufficient to restrict ourselves to instances in which the shortest path between s_i and t_i for every i is at least $\tilde{\Omega}(\sqrt{n})$ long. Otherwise we can greedily route pairs with short paths and remove them. Let OPT be the number of paths in an optimal solution to an instance with only long paths. Since each of the optimal paths is constrained to be long, they share many vertices. In particular, a straight forward averaging argument shows that there must be a vertex u such that $\tilde{\Omega}(\text{OPT}/\sqrt{n})$ paths go through u. We can assume knowledge of u by enumerating over all vertices. How does this information help us? Consider the following variant of the edge disjoint paths problem: find the maximum number of pairs that can be routed via disjoint paths all of which are constrained to go through a specific vertex v. We show that this variant can be well approximated in acyclic graphs because of the following fact: given any vertex v, the set of vertices which can reach v via directed paths and the set of vertices which can be reached by v via a directed paths are *disjoint*. In particular the variant of the edge disjoint paths problem mentioned above induces what we

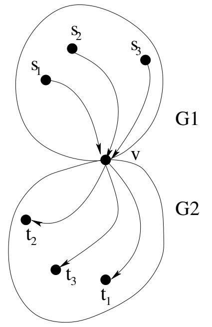

Figure 2: A v-separable instance.

call a v-*separable* instance (to be formally defined shortly). In Section 5 we show that an $O(\log^2 n)$ approximation can be obtained for a v-separable instance via LP rounding. Therefore by guessing u we can route $\tilde{\Omega}(\text{OPT}/\sqrt{n})$ pairs which leads to the promised $\tilde{O}(\sqrt{n})$ approximation. We now define a v-separable instance.

DEFINITION 1. *Let I be an instance of the EDP problem on $G = (V, E)$ with k pairs (s_i, t_i). An instance I is v-separable if we can partition G into two induced graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that:*

- *$V = V_1 \cup V_2$ and $V_1 \cap V_2 = \{v\}$, and*
- *$E = E_1 \cup E_2$, and $E_1 \cap E_2 = \emptyset$, and*
- *for $1 \le i \le k$, $s_i \in V_1$ and $t_i \in V_2$.*

Note that in a v-separable instance all paths from s_i to t_i have to go through v. See Figure 2. Let I be an instance of the EDP on an acyclic graph G. For a vertex $v \in V(G)$, let G_v^{in} (for G into v) be the subgraph of G induced by the vertices of $V(G)$ that can reach v in G. Similarly let G_v^{out} (for G out of v) be the subgraph induced by vertices that can be reached by v in G. Since G is acyclic, G_v^{in} and G_v^{out} share only v. Given a set of pairs (s_i, t_i) to be routed in G, we restrict the pairs to those such that $s_i \in V(G^{in})$ and $t_i \in V(G^{out})$. By doing this we naturally obtain a v-separable instance of the edge disjoint paths problem with $G_1 = G_v^{in}$ and $G_2 = G_v^{out}$. Further we note that any path in G that goes through v is preserved in the new v-separable instance. In Section 5 we show the following result.

THEOREM 4.1. *Given a v-separable instance I of the maximum edge disjoint paths problem along with the partition of G into G_1 and G_2, there is a polynomial time algorithm that finds $\Omega(\mathcal{O}/\log^2 n)$ disjoint paths where $\mathcal{O}$ is the value of any feasible* LP-MCF *solution on I.*

We now formally describe our approximation algorithm for acyclic graphs based on the above theorem. In the algorithm we use a threshold ℓ that will be specified later in the analysis.

Algorithm Acyclic-DP:

1. Let $S = \emptyset$. $X = \{1, 2, \ldots, k\}$.
2. While there exists an i in X and an $s_i \rightsquigarrow t_i$ path P_i such that $|P_i| \le \ell$ do **[Greedy Phase]**
   (a) Add i to S and remove i from X.
   (b) Remove the edges of P_i from the graph.
3. Let G' be the residual graph. max $= 0$, $S' = \emptyset$.
4. For each vertex $v \in G$ do **[v-separable Phase]**
   (a) Approximately solve the v-separable instance induced by v and X in G'. Let S_v be the set of indices of the paths routed through v.
   (b) If (max $< |S_v|$) set max $= |S_v|$ and set $S' = S_v$.
5. Output $S \cup S'$.

There are two phases in the algorithm. Step 2 implements a *greedy* phase in which we remove paths as long as they are *short*, that is of length less than ℓ. S is the set of pairs routed in this phase. Step 4 implements the second phase that solves a v-separable instance for each vertex v induced by v in the residual graph G'. We pick the vertex that results in the most number of paths routed. S' is the set of pairs routed in this phase. We now analyze the performance of the above algorithm.

Let O be the set of pairs routed in some fixed optimal solution. Let O_u be the set of pairs in O but not in $S \cup S'$. We partition O_u into two sets O_1 and O_2 as follows. A pair from O_u is in O_1 if and only if its path in the optimal solution intersects the path of some pair in S. It is easily seen that $|O_1| \le \ell \cdot |S|$ since the path for a pair in S is at most ℓ edges long. Recall from the algorithm description that G' is the residual graph after edges in the paths used to route S are removed. The following claim is easy to see.

CLAIM 1. *All the pairs in O_2 are routed in $E(G')$ and their paths are at least ℓ long.*

Using the above claim and a simple averaging argument, we get the following: there exists a vertex u such that at least $\ell|O_2|/n$ paths from O_2 go through u in G'. Note that $O_2 \subset X$ where X is the set of unrouted pairs at the beginning of Step 4. Hence, in the u-separable instance induced by u and X in G' at least $\ell|O_2|/n$ paths from O_2 can be routed. Therefore, from Theorem 4.1, the algorithm can route $\Omega(\ell|O_2|/(n \log^2 n))$ pairs.

THEOREM 4.2. *There is an $O(\sqrt{n}\log n)$ approximation for the edge disjoint path problem in acyclic graphs. Furthermore, the LP integrality gap is $O(\sqrt{n}\log n)$.*

Proof. From the arguments above, $|O_1| \leq \ell \cdot |S|$ and $|O_2| \leq cn \log^2 n|S'|/\ell$ for some sufficiently large constant c. Choosing ℓ to be $\sqrt{n} \log n$, we obtain that $|O_u| = |O_1| + |O_2| \leq c'\sqrt{n} \log n(|S| + |S'|)$ for some constant c'. Hence we obtain the desired approximation. In the above algorithm the greedy phase is not based on an LP solution. However, it is possible to argue along similar lines with a given feasible LP solution to establish a bound of $O(\sqrt{n} \log n)$ on the LP integrality gap. We omit details in this version. □

5 Approximating edge disjoint paths in a v-separable instance

In this section we prove Theorem 4.1. We use properties of the *single* source flow problem in an important way. We start with a preliminary lemma that guarantees a near optimal feasible solution to LP-MCF with some additional properties.

LEMMA 5.1. *For a given instance I of the EDP let $(x_1, \ldots, x_k, f_{\pi_1}, f_{\pi_2}, \ldots)$ be a feasible fractional solution to* LP-MCF *of value $\mathcal{O} = \sum_i x_i$. There exist absolute constants α and β such that there is a feasible solution $(y_1, \ldots, y_k, g_{\pi_1}, g_{\pi_2}, \ldots)$ to* LP-MCF *such that:*

- *for each i with $y_i > 0$ there is exactly one flow path $\pi(i)$ with $g_{\pi(i)} = y_i = y$ between s_i and t_i,*
- *$y \geq \alpha/\log n$, and*
- *$\mathcal{O}' = \sum_i y_i \geq \beta\mathcal{O}/\log n$ and $\mathcal{O}'$ is an integer.*

Such a solution can be computed in polynomial time.

Proof. Give the feasible solution $(\bar{x}, \bar{f})$ we do flow decomposition and randomized rounding in the standard way. There exists an absolute constant α such that with high probability we get an *integral* solution $(\bar{y}, \bar{g})$ such that the flow on each edge is at most $\log n/\alpha$. Furthermore the value of this solution is at least $\mathcal{O}$. To obtain a feasible fractional solution we scale down the integral solution by a $\log n/\alpha$ factor so that the flow on each edge at most 1. This solution satisfies the requirements of the lemma except for the integrality of $\mathcal{O}'$. The scaling factor can be changed slightly to ensure this property. □

Single source flows: Given a directed (undirected) graph with a single source s and terminals $u_1, \ldots, u_k$ finding the maximum number of disjoint paths from v to the terminals is easy. We can reduce it to the standard source-sink flow by connecting each terminal u_i to a new sink vertex t by a directed edge of capacity 1 and then computing the maximum flow from s to t. An integral flow is guaranteed. Not only is the flow integral, the systems of inequalities describing s-t flow is totally unimodular (see [13, 18]), hence we also obtain decomposition properties. We state the following lemma for our convenience.

LEMMA 5.2. *Let G be a unit-capacity directed graph with a source vertex v and distinct terminals $u_1, u_2, \ldots, u_k$. Let f be a feasible flow vector in which each terminal u_i, $1 \leq i \leq k$ gets a flow of z, $0 \leq z \leq 1$ from v such that kz is an integer. Then there exists an $\ell \leq m$ and for $1 \leq i \leq \ell$, a positive real α_i and an edge disjoint path collection F_i from v to a subset of terminals, with the following properties.*

- *$\sum_{1 \leq i \leq \ell} \alpha_i = 1$, and*
- *the fractional flow $f = \sum_{1 \leq i \leq \ell} \alpha_i F_i$, and*
- *for each terminal u_j, $\sum_{i: u_j \in F_i} \alpha_i = z$.*

Such a decomposition can be computed in polynomial time.

Proof. We connect the terminals to a sink vertex t by directed edges of capacity 1 and then there is an v-t flow of value kz. Let e_i be the edge connecting u_i to t. We can send flow of value kz from v to t where the flow on each e_i is exactly z. Since the flow value is integral and all capacities are 1, this fractional v-t flow can be decomposed into a convex combination of *integral flows*, each of value kz, such that each edge e that is fractionally used to capacity x_e is used in the convex combination to x_e. Each integral flow from v to t of value kz defines a disjoint path collection from v to t and hence from v to the terminals. We put u_i in the path collection corresponding to an integral flow if e_i is used in that flow. The lemma follows. □

Our algorithm for a v-separable instance is based on the following observation that establishes the connection to single source flows.

PROPOSITION 5.1. *Let $\bar{x}$ be a feasible solution to* LP-MCF *for a v-separable instance I. Then it defines a feasible solution to two separate single source flow problems: I_1 with v and $s_1, s_2, \ldots, s_k$ in G_1, and I_2 with v and $t_1, t_2, \ldots, t_k$ in G_2. The flow value in I_1 from s_i to v and in I_2 from v to t_i is x_i.*

Algorithm v-sep:

1. Solve LP-MCF on I to obtain a fractional solution $(\bar{x}, \bar{f})$ with value $\mathcal{O}$.
2. Convert $(\bar{x}, \bar{f})$ to a solution $(\bar{y}, \bar{g})$ with value $\mathcal{O}'$ and the properties described in Lemma 5.1. Let y be the common flow value for the pairs that have positive flow.
3. Let $\bar{z}$ and and $\bar{w}$ be the feasible fractional solutions to the single source flow problems induced by $(\bar{y}, \bar{g})$ in G_1 and G_2 respectively. Decompose the single source flow in each instance separately to obtain convex combinations $(\bar{\alpha}, \bar{F})$ and $(\bar{\beta}, \bar{H})$ of disjoint path collections, as guaranteed by Lemma 5.2.

4. From $\bar{F}$ pick a random disjoint path collection F_a according to the distribution $\bar{\alpha}$. Independently, from $\bar{H}$, pick a random path collection H_b according to the distribution $\bar{\beta}$.

5. If s_i is connected to v in F_a and v is connected to t_i in H_b, connect s_i and t_i by joining the the two paths into a single path. Otherwise the pair (s_i, t_i) is not connected.

Since $F_a \subseteq E_1$ and $H_b \subseteq E_2$ and $E_1 \cap E_2 = \emptyset$, it is easy to see that a feasible solution is generated for the original instance I.

LEMMA 5.3. *The probability that a pair (s_i, t_i) with positive flow y in the fractional solution $(\bar{y}, \bar{g})$ is connected by a disjoint path is y^2.*

Proof. Consider the convex decomposition $(\bar{\alpha}, \bar{F})$ in the single source instance in G_1. We pick a random disjoint path collection from the decomposition. Lemma 5.2 guarantees that s_i will be in the disjoint path collection with probability y, the value of the flow to v. By exactly the same reasoning, v will be connected to t_i with probability y. Since the two events are independent, with probability y^2, s_i and t_i will be connected by a disjoint path via v. $\Box$

Hence the expected number of pairs that will be connected is $\sum_{i,y_i>0} y^2$. However, $\mathcal{O}' = \sum_{i,y_i>0} y$, and from Lemma 5.1, $\mathcal{O}' \geq \mathcal{O}/O(\log n)$ and $y = \Omega(1/\log n)$. Therefore the expected number of pairs that are connected is $\mathcal{O}/O(\log^2 n)$. $\mathcal{O}$ is an upper bound on the integer optimal solution and this completes the proof of Theorem 4.1. We obtain the following corollary of Theorem 4.1.

COROLLARY 5.1. *Let G be a (directed) undirected graph with pairs (s_i, t_i), $1 \leq i \leq k$ such that there exist edge disjoint paths for ℓ of the k pairs all of which go through a specified vertex v. We can find in polynomial time a set of $S \subset \{1, 2, \ldots, k\}$ such that:*

- *$|S| = \Omega(1/\log^2 n)\ell$, and*
- *for each $i \in S$ there exist disjoint paths P_i and P'_i s.t P_i connects s_i to v and P'_i connects v to t_i, and*
- *for all $i, j \in S$, $i \neq j$, the paths P_i and P_j are edge disjoint and P'_i and P'_j are edge disjoint.*

6 Unsplittable Flow

The *unsplittable flow problem* (UFP) is a generalization of the maximum edge disjoint path problem. We are given an edge-capacitated graph $G = (V, E)$ with c_e denoting the capacity of edge $e \in E$. Each pair (s_i, t_i) also specifies a demand ρ_i which needs to be routed on a single path. The objective is to find the maximum number of pairs that can be routed without violating edge capacities. A special case of UFP that is of interest is the uniform capacity UFP (UCUFP for short). In UCUFP all edges are assumed to have the same capacity C. Note that this problem generalizes EDP, and hence is at least as hard. Until now, the best known approximation ratio for UCUFP was $O(\sqrt{m})$ [19, 2]. We now observe that the approximability bounds for EDP obtained in Theorems 3.3 and 4.2 apply to UCUFP as well. Given an arbitrary set of demands, we can partition it into two sets S_1 and S_2 where S_1 contains demands at most as large as $C/2$ and S_2 contains the remaining demands. Baveja and Srinivasan [2], and Azar and Regev [1] give an $O(\sqrt{n})$-approximation for UCUFP when each demand is at most half the edge capacity. Thus we can get an $O(\sqrt{n})$-approximation with respect to demands in S_1. The demands in S_2, on the other hand, cannot share a common edge and hence define an instance of EDP. Thus we can approximate the demands in S_2 as well as given by the bounds in Theorems 3.3 and 4.2. The result follows by taking better of the two solutions.

7 Conclusions

We can extend the algorithm Acyclic-DP from Section 4 to general graphs. Acyclicity of the graph is used in Step 4 to argue that the flow paths that go through the vertex with maximum congestion (for that matter any vertex) induce a v-separable instance. This allows us to argue that the paths routed are disjoint. For general graphs we use Corollary 5.1 in place of Theorem 4.1. The theorem below follows from exactly the same reasoning and analysis as in Section 4 but we are no longer guaranteed edge disjoint paths, however Corollary 5.1 guarantees paths of congestion at most 2.

THEOREM 7.1. *For any instance of the EDP problem there is a polynomial time algorithm that connects $\Omega(\text{OPT}/(\sqrt{n}\log n))$ pairs with paths that have congestion at most 2 and there exists a special vertex v such that all intersecting paths in the solution go through v. Furthermore, suppose P_i that connects the pair (s_i, t_i) intersects with P_j that connects the pair (s_j, t_j), then the segments $s_i \rightsquigarrow v$ of P_i and $s_j \rightsquigarrow v$ of P_j are edge disjoint, and the segments $v \rightsquigarrow t_i$ of P_i and $v \rightsquigarrow t_j$ of P_j are edge disjoint.*

We observe that the paths guaranteed by the theorem above, though intersecting, have substantial structure. Our theorem is mainly interesting from that perspective, in fact as we mentioned earlier an $O(\sqrt{n})$ bound with congestion 2 follows from randomized rounding [14]. We believe that the restricted nature of the paths can be taken advantage to obtain improved bounds for *undirected* graphs. To illustrate our intuition we consider the graph shown in Figure 3. The solid lines represent the well known *brickwall* graph. It is easy to see that in the brickwall at most one of the pairs (s_i, t_i) can be routed using edge disjoint paths, while *all* pairs can be routed if congestion 2 is allowed. Hence, the ratio between

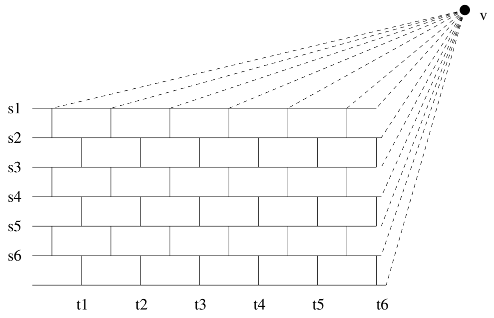

Figure 3: Solid lines indicate the brickwall graph.

the number of pairs that can be routed with congestion 2 allowed and the number with congestion 1 allowed, can be very large.

Now consider the graph in Fig 3 where a vertex v is attached with edges shown by dotted lines to the brickwall. In this graph all pairs (s_i, t_i) can be connected via paths that are restricted to be of the form specified in Theorem 7.1. The paths for pair (s_i, t_i) uses the edges along the row of s_i to reach v and then edges down the columns to reach t_i. However, in this case, we can actually route half the pairs using edge disjoint paths! We also note that we can route half the pairs for any permutation of the t_i at the bottom of the grid. The property that all paths go through v enables efficient uncrossing. We give a lemma that indicates the consequence of understanding this problem.

LEMMA 7.1. *Let $P_1, P_2, \ldots, P_k$ be k paths in an undirected graph such that for each i $P_i = s_i \rightsquigarrow v \rightsquigarrow t_i$ and no more than two paths use any edge in the graph. Furthermore, for distinct i and j, the segments $s_i \rightsquigarrow v$ of P_i and $s_j \rightsquigarrow v$ of P_j are disjoint and the segments $v \rightsquigarrow t_i$ of P_i and $v \rightsquigarrow t_j$ of P_j are edge disjoint. Given any such instance, if at least $\Omega(k^{1-\varepsilon})$ of the pairs can be routed via disjoint paths then there is an $\tilde{O}(n^{(1+\varepsilon)/(2+\varepsilon)})$-approximation for the maximum edge disjoint path problem in undirected graphs.*

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References

[1] Y. AZAR AND O. REGEV. Strongly polynomial algorithms for the unsplittable flow problem. In *Proceedings of 8th IPCO*, 2001.

[2] A. BAVEJA AND A. SRINIVASAN. Approximation algorithms for disjoint paths and related routing and packing problems. *Mathematics of Operations Research*, Vol. 25, pp. 255–280, 2000.

[3] J. CHERIYAN, H. KARLOFF, AND Y. RABANI. Approximating directed multicuts. In *Proceedings of 42nd FOCS*, pp. 320–328, 2001.

[4] S. EVEN AND R. E. TARJAN. Network flow and testing graph connectivity. *SIAM Journal on Computing*, Vol. 4, 1975, pp. 507-518.

[5] S. FORTUNE, J. HOPCROFT AND J. WYLLIE. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, Vol. 10, No. 2 (1980), pp. 111–121.

[6] A. FRANK. Packing paths, cuts, and circuits - a survey. In B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, eds., *Paths, Flows and VLSI-Layout*, 49–100. Springer Verlag, Berlin, 1990.

[7] M. R. GAREY AND D. S. JOHNSON. *Computers and Intractability: A Guide to the Theory of NP-completeness*. Freeman, 1979.

[8] N. GARG AND J. KONEMANN. Faster and Simpler Algorithms for Multicommodity Flow and other Fractional Packing Problems. In *Proceedings of 39th FOCS*, 1998.

[9] N. GARG, V. VAZIRANI AND M. YANNAKAKIS. Primal-dual approximation algorithms for integral flow and multicut in trees. *Algorithmica*, 18 (1997), pp. 3–20. (Preliminary version in *Proceedings of 20th International Colloquium on Automata, Languages, and Programming*, 1993, pp. 64–75.)

[10] V. GURUSWAMI, S. KHANNA, B. SHEPHERD, R. RAJARAMAN, AND M. YANNAKAKIS. Near-Optimal Hardness Results and Approximation Algorithms for Edge-Disjoint Paths and Related Problems. In *Proceedings of 31st STOC*, pp. 19–28, 1999.

[11] J. M. KLEINBERG. *Approximation algorithms for disjoint paths problems*. PhD thesis, MIT, Cambridge, MA, May 1996.

[12] S. G. KOLLIOPOULOS AND C. STEIN. Approximating disjoint-path problems using greedy algorithms and Packing Integer Programs. *Integer programming and Combinatorial Optimization*, 1998.

[13] E. L. LAWLER. Combinatorial Optimization: Networks and Matroids. Holt, Rinehart and Winston, 1976.

[14] T. LEIGHTON, S. RAO, AND A. SRINIVASAN. Multicommodity flow and circuit switching. In Proceedings of the Hawaii International Conference on System Sciences (HICSS), pages 459-465, 1998.

[15] B. MA AND L. WANG. On the inapproximability of disjoint paths and minimum steiner forest with bandwidth constraints. *Journal of Computer and Systems Sciences*, vol 60, pages 1–12, 2000.

[16] P. RAGHAVAN AND C. D. THOMPSON. Randomized rounding: A technique for provably good algorithms and algorithmic proofs. *Combinatorica*, Vol. 7 (1987), pp. 365–374.

[17] N. ROBERTSON AND P. D. SEYMOUR. Outline of a disjoint paths algorithm. In B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, Eds., *Paths, Flows and VLSI-Layout*. Springer-Verlag, Berlin, 1990.

[18] A. SCHRIJVER. Theory of Linear and Integer Programming. John Wiley and Sons, 1986.

[19] A. SRINIVASAN. Improved approximations for edge-disjoint paths, unsplittable flow, and related routing problems. *Proc. of the 38th Symposium on the Foundations of Computer Science*, 1997, pp. 416–425.

[20] D. B. WEST. Introduction to Graph Theory. Prentice-Hall, Inc., 1996.